Calculus

and

Computer-supported Collaborative Learning

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Abstract

Success in mathematics is the gateway to many careers in the sciences, and increasingly in other fields such as economics and commerce. Ubiquitous high failure rates in Calculus courses prevent capable students from pursuing their career goals. Hence educators seek improvements in instructional design that will help more students to succeed. The use of technology in mathematics education is often seen as a potential source of improvement in student learning of mathematics. We studied two cases of integration of technology in Calculus I courses: WebCal and Maple. In both studies we used a 2 x 2 factorial design to assess the impact of both collaborative learning and the use of technology on student achievement and changes in student motivation to study mathematics. In the course of this research we developed measures for assessing student knowledge of Calculus I: arithmetic/algebraic skills; use of symbolic language; understanding of algorithms; correct answer; understanding of graphs; and conceptual understanding. There were 384 students and six instructors participating in this study. In the WebCal study we determined that usage of WebCal had a positive impact on understanding of algorithms and graphs. There were no significant differences found in the Maple study. Interestingly, WebCal students in lecture sections outperformed WebCal students in collaborative sections on arithmetic/algebraic skills and on use of symbolic language. On the other hand, only collaborative learning had a positive impact on students' ability to solve problems without making errors. There were no significant differences in changes of student motivation to study mathematics found in the WebCal study. In contrast, student motivation significantly decreased in the Maple study. In both studies, we found that students using the technology reported that they studied longer hours in comparison to their counterparts in classes where technology was not used.

Introduction

In the twenty-first century computer technology will be ubiquitous, and self-directed learning will be the keystone of both personal and marketplace success, thus it is imperative that we prepare our students accordingly (Sauer, 1990). In recognition of this need the new CÉGEP science program requires integration of computer technology into the mathematics and science curriculum. Because success in mathematics is the gateway to many careers in the sciences, and increasingly in other fields such as economics and commerce, high failure rates in mathematics at Vanier and across the Reseau (and also in the U.S.A., Ferrini-Mundy & Lauten, 1994) are potentially disastrous. Thus, every one concerned hopes that the new program will not only be effective in bringing about the integration of computer technology, but also at the same time will improve the overall success of mathematics instruction. Improved achievement in mathematics would help reduce the shortage of science graduates in Quebec and in Canada (Baillargeon, Demers, Ducharme, Foucault, Lavigne, Lespérance, Lavallée, Ristic, Sylvain, Vigneault, 2001, Nankivell, 1998) relative to other developed countries, and thereby decrease the disadvantage Quebec and Canada have in comparison to their major trade competitors.

We note that efforts to improve student conceptual understanding are being actively pursued within the elementary and secondary mathematics education sectors in Quebec. These efforts involve implementation of a constructivist educational philosophy and often a collaborative learning[1] (CL) approach as well. Furthermore, the curriculum of mathematics courses in Secondary IV and V demands the integration of technology in terms of the active use of graphing calculators. A cohort of 2242 students graduating in the summer of 2003 and enrolling in four public anglophone CÉGEPs was asked to fill in a questionnaire that they never used the graphing calculators. In addition, 970 (46% of those wh regarding their experience in using calculators (Rosenfield, 2004). Of the 2108 students who responded to the items pertaining to use of Calculators, 527 (25% of all) students reported o had used graphing calculators) reported that they had not been taught how to use graphing calculators by their teachers. These results indicate that the integration of technology may not be proceeding as it was envisaged by reformers within the MEQ. On the other hand, a review (Barton, S., 2000) of fifty two studies concerning the effect of the integration of graphing calculators into course work determined that more than two-thirds of the studies reported greater achievement in the classes using either graphing calculators or computer algebra systems (CAS) over the classes that did not involve the use of technology. Moreover, 75% of studies reported better outcomes on measures of conceptual understanding in classes using either graphing calculators or CAS over classes that did not use either technology. It is important to note that although these results are promising, many studies indicate that there was no significant difference. This suggests that while the use of technology can

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[1]The terms "cooperative learning" and "collaborative learning" are used synonymously in this report.

have a positive effect, perhaps another factor, such as how technology is used, may be equally important.

CÉGEP mathematics instructors have not undertaken to reform Calculus teaching in a manner similar to the changes taking place in Secondary School Mathematics. Un fortunately, students taking standard Calculus courses have been shown not to achieve expert-like knowledge (Roddick,1995). Thus, in developing WebCal materials our goal was to design a course from which students would graduate having acquired expert-like knowledge of the central concepts of Calculus, and that this knowledge would be transferable to further studies in mathematics, and where appropriate to other disciplines as well (Fisher, 1996). However, there was and still is insufficient evidence that integrating computer technology, collaborative learning, and a constructivist approach leads to improved achievement and/or conceptual understanding of mathematics at the CÉGEP level. During the 1999-2000 and 2000-2001academic years, while still developing and refining our materials, we carried out observations in one WebCal-based course implementation. This qualitative study (Dedic, H., Rosenfield, S., Cooper, M. & Fuchs, M., 2001) suggested that while some students did demonstrate the sound conceptual understanding of concepts of Calculus that we hoped for, the effectiveness of this instructional setting seemed highly dependent on the characteristics of the student. In this context, effectiveness represents both an understanding of Calculus concepts and student motivation to study mathematics. Thus, the issue became one of how we could gather quantitative evidence examining the integration of computer technology into the teaching/learning of Calculus. Another central objective was to differentiate between the impact of group work and the integration of technology on both the promotion of conceptual understanding and student motivation to study mathematics.

Computer technology can be integrated into the teaching of mathematics in a number of ways. For example, many instructors provide their lecture notes online. The focus of such efforts is to facilitate access to information. Use of a course-management package, such as Blackboard or WebCT, not only facilitates access to course materials, but also promotes student-student and/or student-instructor online conversations. The focus of such efforts is thus to provide access to both a source of knowledge, and also to provide help when needed. An entirely different mode of integration of computer technology involves the use of computer algebra systems (CAS) such as Maple or Mathematica. The focus of these efforts is often to teach students how to use the tools that mathematicians and scientists currently use in their work. Mathematicians use CAS systems because they remove the tedium of routine computation. Instructors use these systems to speed up the computations, and, therefore, allow students to explore mathematical concepts. Lastly, sometimes the integration of computer technology in mathematics involves the use of "drill and practice" packages (*e.g.*, WebWorks). The focus of such efforts is to provide students with instant feedback regarding their work.

Prompted by the desire to provide quantitative evidence of the effectiveness of the integration of computer technology, and by the variety of possible uses of technology in mathematics instruction, we decided to investigate the effectiveness of two different modes of such integration: a course that includes access to web-based course materials and includes the use of a CAS package (WebCal); and, a course that combines the use of a CAS package with a course-work approach that focuses on teaching how to use the CAS in exploring and "doing mathematics". We also decided to examine the impact of instantaneous feedback on student learning in a course with web-based materials.

This final report begins below with the theoretical framework that guided the investigation. Next there is a general description of the methodology used, followed by a detailed description of the protocol used, and then the results obtained in two experiments: the effectiveness of a web-based course (face-to-face delivery, not distance education), WebCal, relative to a standard instructional design; the effectiveness of a Maple-based approach relative to a standard instructional design. Conclusions and recommendations flowing from this research then sum up this report. Since we have already published the results of the study concerning the effectiveness of simple vs multiple-try instantaneous feedback in a web-based course, WebCal, we have appended those materials intact below the conclusions concerning the two other studies.

Theoretical Framework

There are four theoretical perspectives that guide this research: the Calculus Reform Movement (Hodgson, 1987); the tenets of a constructivist paradigm, as well as the work of several learning theorists (Piaget, 1954, Ausubel 1963, Posner, Strike, Hewson, & Gertzog,1982, Entwistle & Tait, 1996); the theory of the impact of motivation and self-regulation on student learning (Pintrich, Marx & Boyle, 1993, Zimmermann, 1990); and, the theory of collaborative learning. In addition, this research was guided by our earlier qualitative study of student perceptions while taking a Calculus course with web-based materials (Dedic et al., 2001).

Reform Calculus:

WebCal, our web-based course, complies with the "Rule of Four" of Reform Calculus whereby the concepts of Calculus are presented using a balance of verbal, graphical, numerical and algebraic perspectives. It has been shown that expert mathematicians move fluidly between these four perspectives in solving problems (Hodgson, 1987). Including the "Rule of Four" approach in Calculus instruction may lead to students developing a deeper, more expert-like understanding of the central concepts of Calculus and higher rates of student success (Ferrini-Mundy et. al., 1994). In addition, the balance of analytical, graphical, numerical and verbal perspectives of mathematical offers more entry points to understanding for students with different learning styles (Armstrong & Hendrix, 1999). Unfortunately, for many students the details involved in shifting perspectives - shifting from an analytical to a graphical or numerical representation of a function involves generating a table of numerical values, using the table to generate a graph - are tasks sufficiently complex that, even if they choose to do these tasks, many students do not have the cognitive capacity to then simultaneously engage in conceptualization. CAS systems effortlessly compute tables of values and generate an associated graph when the student merely types in the formula. This permits students to focus their attention on numerical trends in the table, or on features of graphs as related to features in the formulae. Thus, the hope is that using computers in this fashion might enable students to grasp difficult concepts of Calculus because it allows them to focus on conceptualization, instead of on the mechanics involved in generating numerical, graphical and symbolic perspectives (Stephens & Konvalina, 1999).

Learning theories and constructivism:

Learning theorists distinguish two different processes by which learners acquire new concepts, each of which results in a different knowledge structure. Certain cognitive processes which result in expert-like knowledge have been labelled meaningful learning (Ausubel, 1963), deep processing (Entwistle. et. al. 1996) or the process of conceptual change (Posner. et. al. 1982). On the other hand, processes which result in a knowledge structure that leaves the learner ill-equipped to use it in problem solving or to transfer it to different fields have been called rote learning (Ausubel, 1963) or surface processing (Entwistle. et. al. 1996). The process of conceptual change (our choice of label) is accomplished by learners when cognitively elaborating new information by relating it to prior knowledge in various ways: relating different perspectives of concepts; engaging in the process of reconciliation of previously held views with data gained through their own observation and experimentation; formulating and testing hypotheses; questioning and formulating conclusions, etc. However, rote learning (our choice of label) is accomplished by memorizing verbatim definitions of new concepts and memorizing algorithms for solving problems rather than by elaborating links between information and concepts (Mazur, 1996).

Theorists (Posner. et. al. 1982) use the term "conceptual change" to describe a learning process that promotes students acquiring a conceptual understanding that allows them to effectively use the concept in novel problem solving ((Posner. et. al. 1982); Posner & Strike, 1992; Chinn and Brewer, 1993; Dykstra, Boyle & Monarch, 1992; Redish, 1994). According to these researchers instructional designs rooted in this theory share four common characteristics: creating dissatisfaction in students with their prior understanding; providing a convincing demonstration that new concepts are understandable; showing that the new concept is plausible in view of student background knowledge; and, showing that the new concept is more useful in problem solving than the old concept.

Proponents of the constructivist paradigm believe that a learner's active engagement with material favours the process of conceptual change, while a learner's passive listening to lectures providing new information tends to lead to rote learning. It is important to note that just the use of computers in the classroom in itself enhances the interaction of students with materials, an effect which by itself has been shown to promote achievement of students (Mayer, 1997). However, Osta (1994) points out that there are gender based differences in the reaction of students to learning situations involving computers.

Papert, (1980, 1992) showed that children construct a new mental model (elsewhere called undergoing conceptual change or building conceptual understanding) by actively experiencing sufficient examples. His excitement in both books arises from his studies wherein computers have been used to construct "Microworlds" that allow each child to rapidly generate, what is for them, sufficient experiences to enable conceptual change. Papert (1992) noted that computer generated experiences allow children to become abstract thinkers at a very early age. Similarly, in mathematics each student needs to experience a sufficient number of instances of a given phenomenon before hypotheses can be formulated, tested and conclusions drawn. For example, understanding the connection between the formula for a given function and the shape of its graph requires exposure, in a systematic manner, to many formulae and their corresponding graphs. In a standard lecture-based course, the task of generating repeated examples is either neglected completely because it is tiresome and time consuming for the teacher, or left to each student as a paper and pencil task. Most students, faced with the tiresome effort of performing many such tasks on their own, fail to complete sufficient examples to allow them to grasp the underlying concept. Instead, they choose rote memorization of superficial patterns as the most energy-efficient learning method. Use of CAS systems allows students to rapidly generate as many instances of a given phenomenon as they need to see the pattern, and hence to construct their understanding. Thus, the use of computers may enable many students to learn through experimenting, and then to develop a conceptual understanding otherwise achieved only by the exceptional few.

An essential part of learning is the development by each student of an understanding of the cycle often called the scientific method (observations, leading to a belief about the structure governing a particular phenomenon, testing of that belief through additional observations, and either formation of new beliefs or further generalization of the old beliefs). Computer-mediated instruction in science and mathematics provide learners with visual feedback enabling students to travel through this cycle of discovery (Papert, 1980). When working with computers students may encounter dissatisfaction with previously held views because their predictions are immediately shown to be invalid. By removing much tedious mechanical work, the technology allows a large number of instances to be generated within a short time period, and thus can rapidly lead students to become dissatisfied with their previously held views. This is one of the four cognitive conditions necessary for conceptual change to occur (Posner et. al., 1982). Similarly, technology allows students to experiment and to rapidly generate many instances of the application of a new concept. Such experiments provide evidence of the fruitfulness of the new concept in problem solving. In addition, the new concept becomes an intelligible and plausible alternative to previously held views. Consequently, the remaining three cognitive conditions prerequisite to conceptual change can also be satisfied when a student is exposed to computer-mediated instruction.

Motivation.

Pintrich, Marx and Boyle (1993) introduced the concept of hot conceptual change. They proposed that student conceptual change does not depend solely on the four cognitive conditions proposed by Posner et al. (1982), but also depends on student characteristics and beliefs (*e.g.*, prior knowledge, motivation, epistemological beliefs as applied to the discipline, locus of control, self-regulation). Most, if not all, students learning Calculus need to expend a substantial effort. Motivation to engage in this effort comes not from objective reality but from our subjective interpretation of reality. Expectancy-Value Theory (Tolman, 1932; Rotter, 1966; Rotter, Chance & Phares,1972) seems to offer a good perspective on motivation in this context. For example, students who do not value knowledge of mathematics because it does not seem to be important in their lives, or because they have a negative affect towards it (Lafortune, 1992), are unlikely to truly engage in the learning process. Consequently, they are unlikely to achieve conceptual understanding. Furthermore, if students have a low subjective expectation concerning their probability of success, then they are unlikely expend the effort that is necessary to achieve conceptual understanding. Thus, students who have a low self-efficacy in mathematics (Bandura, 1986) may not engage in the learning process or may not persist in the face of adversity when learning difficult to grasp concepts.

Another motivational variable is often implicated in a student's efforts to achieve conceptual change. More effort is expended by mastery oriented students than by performance oriented students (Dweck & Elliot, 1988). Furthermore, students' beliefs about knowledge have been shown to impact on learning. Students who believe that knowledge is innate or acquired quickly are unlikely to patiently struggle to understand difficult concepts (Schommer, 1990).

Pintrich et al. (1993) point out that characteristics of an instructional setting (*e.g.* task and evaluation structures, course management, interactive engagement, feedback) influence whether conceptual change occurs. For example, assessments that are formative in nature, as opposed to summative, encourage students to continue to develop meaningful understanding (Saunders,1992). Classroom environments that foster mastery goals are also likely to impact positively on student conceptual change (Ames, 1992). It has been shown that feedback effectively promotes meaningful learning, given that the instructional setting provides both the tasks from which they can draw data and incorporates feedback structures that allow students to gauge their performance (Lou, Dedic & Rosenfield, 2002).

Collaborative learning

Many researchers (Abrami et al., 1995; Bosse & Nandakumar, 1998; Brophy, 1995; Scardamalia, Bereiter, Brett, Burtis, Calhoun & Lea, 1992; Lafortune,1998 and Lafortune et. al., 1994)) demonstrated that collaborative learning instructional settings promote student achievement and conceptual change by requiring students to engage in conversations concerning the subject matter. The construction of meaning is also enhanced by the need to present a well elaborated idea to the group (Harasim, 1987), to defend ideas against criticism, and in turn, to criticize the ideas of others. In addition, students become actively involved (Hake, 1998) and therefore, their individual conceptual understanding (or misunderstanding) is debated and clarified by their peers. Consequently, students in these collaborative environments are more likely to take ownership of their ideas and make a conceptual change.

Qualitative study of web-based Calculus course

A study funded by SSHRC was conducted in the Fall semester of 1999 with 44 students enrolled in a web-based Calculus course (Dedic, et al., 2002). Classes were held in computer labs. Initially, each student had a computer and, although they were encouraged to work in pairs, such behaviour was not evident. In the fifth week the class moved to the then newly-completed electronic classroom, where pairs of students shared a single computer, and collaborative behaviour began to spread. The teacher provided assistance to groups of students as requested, or lectured to the whole class when many students seemed to be asking the same question. Since this study was exploratory in its objectives, unstructured weekly interviews with eight students were used to gather data on the satisfaction of students with the course, as well as on their subjective evaluation of their learning in the new instructional setting. In addition, more specifically, the interviewer focussed on the perception of students regarding the impact of interactive exercises on their understanding of the concepts of Calculus.

We gathered anecdotal data from students in our web-based course that indicated that they performed well on conceptual questions, on questions which require students to use all four perspectives of mathematical concepts flexibly, as well as on questions which require students to demonstrate the ability to use problem-solving algorithms. We also discovered that students mastered some concepts faster than in a regular class, *e.g.*, they learned the concept and computational techniques of limits in two weeks, instead of the usual three to four weeks. We hypothesized that this is due to these students having formulated a rich conceptual structure in this instructional setting. Thus, it became necessary to test this hypothesis quantitatively, and study whether a web-based but face to face instructional setting, both independently and interacting with student characteristics, impacts differently on achievement and conceptual understanding than a regular Calculus class setting does.

We also found that student engagement rose dramatically after the move into a physical setting that promoted collaborative work. All the students that were interviewed commented on the positive impact of collaboration on their motivation and learning. This finding corroborates the findings of many researchers (*e.g.,* Abrami et al. 1995) that collaborative learning instructional settings promote student achievement and conceptual change by requiring students to engage in conversations about subject matter. All students also commented on the merits of the collaborative quizzes that the instructor used. The quiz questions were complex but students could use the materials on the web and collaborate in arriving at solutions. Without exception, students claimed that the quizzes helped them to develop a better understanding of concepts. These activities not only promoted conceptual change but also enhanced subjective perceptions of learning and promoted student motivation.

In addition we observed that student prior motivation, epistemological beliefs about mathematics and affect towards mathematics interacted with our web-based instructional setting in determining how thye approached learning tasks. Some students felt that the use of technology was an additional burden for them. For example, one student complained that in comparison with his peers in traditional sections of Calculus, his tasks were more time-consuming. Some students questioned whether the graphical and numerical approach is useful in the developing of their understanding of Calculus since they knew that their peers in regular classrooms did not use this approach. Consequently, they were asking why they should use the computers to facilitate "redundant" tasks. Although one complainant admitted that he had probably developed a deeper understanding of the subject than his friends, and had excellent grades, he persisted in voicing doubts whether it was worth the effort. Another student persisted in his negative attitude towards mathematics, and, eventually, failed the course. On the other hand, several interviewed students gradually changed their attitude towards the learning tasks, talked about having fun experimenting and learning in this context. We noted that they exhibited the typical behaviours of motivated students (*e.g.,* perseverance) in class and obtained an indication that motivational characteristics (*e.g.,* valuing of mathematics, affect towards mathematics, perseverance) interacted with the instructional setting and so we saw the need to explore this phenomenon quantitatively.

We also observed that some students were overwhelmed by the computer interface. Often, the same students also reported a limited prior use of computers. In addition, many of these students only acquired internet access at home during the course. It appears that for such students we removed one burden, multiple and repetitive mechanical tasks of graphing and computing values, and replaced that burden with a new one, equally frustrating and time-consuming. It is important to note that this experiment was carried out in the Fall 1999. Subsequent rapid change in access to both computer technology and the internet may have minimized this problem in Fall of 2001 and 2002.

Some students were dissatisfied in our web-based instructional setting and did not engage in experiments, because they believed that it was the role of the teacher to dispense knowledge. We propose that their behaviour is related to their epistemological belief that knowledge is certain. To use an analogy, some students perceive learning as a process similar to the process of getting information off the web, that is search, download, and store the facts on the local hard disk. Others see it as a creative process. However, we observed that these epistemological beliefs may change. One student reported such a change over the course of the semester and then said " What's surprising with this semester, is that before [*in the*] first semester[2] I [*would*] get a good grade, the second semester I [*would*] get a good grade but let's say you ask me something from the first semester I don't know anything about it. I forget everything. What's surprising here is that I remember . . . That means somehow I got a different education. That means I'm not really worrying about the test." This anecdotal testimony indicates that this student understood the concepts since he could remember them, which is in agreement with Hammer's (1994) contention that the epistemological beliefs of a student and the likelihood of conceptual change taking places are strongly related. Furthermore, it implies that web-based settings may result in students changing their

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[2]The student was referring to the two semesters in his high school 536 course.

beliefs, while it has been shown that standard science education settings reinforce beliefs that knowledge is certain (Paulsen & Wells, 1998).

The above quote also indicates that this student's self-efficacy was positively affected in this instructional setting. On the other hand, we have also noted that one student reported a decrease in confidence. Although she saw herself as a very motivated and internally driven student, who had always been good in mathematics, she stated that she often felt unsure whether, as she put it, she was "seeing the right things" when drawing conclusions from experiments. She acknowledged that she was doing well in the class despite feeling unsure, but in her words, "I am learning Calculus by myself", and that although this fact ". . . should make me feel confident, . . . it does not. I am never sure ...". Obviously, this instructional setting was not having the positive effect on her feelings of self-efficacy that we had imagined inquiry-learning would evoke.

Our instructional design was initially intended as a student-centred learning environment with assistance provided by the teacher as needed. Student satisfaction varied from enthusiastically positive to emphatically negative. We suspect that student satisfaction may be related to locus of control, with dissatisfied students being externally controlled learners. Students who craved more step-by-step instruction were also amongst those who complained most about the new environment, although, within the group who were interviewed, dissatisfaction had no impact on achievement. Our findings are at odds with results reported by Davies & Berrow (1998) who demonstrated that externally controlled students did less well in a computer-supported environment. In response to student dissatisfaction, the teacher introduced lectures which included modelling how to experiment. This led to an increase in student satisfaction, engagement and self-regulation in some students. Since the aim is for students to become self-directed independent learners, we see the need to systematically study the change in student self-regulation and achievement in a scaffolded-modelling condition, *i.e.*, where teacher modelling is gradually reduced and the learning environment becomes increasingly student-directed. We need to investigate the interaction of locus of control, use of self-regulatory strategies, achievement and motivation in each of the different instructional settings.

We note that the notion of using experimentation in learning, mirroring the scientific method, is also recommended as a useful technique for constructing conceptual understanding by Barbeau, Montini & Roy (1997). However, our own preliminary results (Dedic, et al., 2002) have shown that simply providing students with the opportunity (time in class to work on a task) and tools that make experimentation possible (in terms of the time, energy and mental capacity required) is not a sufficient condition to impel most of them to actually do it, and hence improve their grasp of concepts in Calculus. Some students failed to see any patterns, became frustrated and reverted to memorization. We hypothesize that students in this setting also need to be self-regulated learners to make the most of it; that is, they must possess appropriate meta-cognitive strategies (*e.g.*, evaluation of one's own knowledge) and appropriate cognitive strategies (*e.g.*, generalization), as well as understanding the processes of experimentation.

Collaborative learning is an important component of this implementation. Our preliminary results (Dedic, et al., 2002) show a positive impact of student collaboration on affect towards mathematics, motivation and conceptual understanding. The use of technology can facilitate communication and interaction amongst students outside the classroom. Thus, using technology can extend student collaboration, which in turn has been shown to promote both traditional achievement outcomes and conceptual understanding. Our results seem to corroborate those of Neff (1998), who indicates that the use of technology (CAS and electronic communication), particularly in Calculus and Physics courses, greatly increases active participation and correspondingly diminishes failure rates.

Distance education (DE), is now making inroads in day school, with as much as sixty percent of enrollment in some DE courses consisting of full-time day students (Johnson, 1998). This would seem to indicate quite strongly the need amongst daytime students for courses in which they can exert more control over both pace and sequencing. Thus, placing resource material on the Web satisfies the needs of these students to self-direct their learning.

Hypotheses

Our previous work, which was exploratory and anecdotal in nature, uncovered interesting processes. The natural next step was a quantitative investigation of those processes. Thus, the primary focus of this study was to contrast the effectiveness of two different computer-supported instructional settings in Calculus with or without collaborative learning against the effectiveness of standard instructional settings with or without collaborative learning. We address the following questions: "Does incorporating computer and communication technology into the teaching of Calculus generate a positive effect in terms of higher student scores on measures of learning and student motivation, and can we separate any such effect from that obtained by incorporating collaborative learning structures?"

Thus the hypotheses in this study can be stated as follows:

1. Comparison of the performance of students using WebCal versus students not using computers at all:
    a. Students using WebCal will outperform students not using WebCal.
    b. Students in Collaborative Mode of Instruction sections will outperform students in Lecture Mode of Instruction sections.
    c. There will be a positive interaction between the impact of using WebCal and being in a Collaborative Mode of Instruction section on student performance.
    d. Students using WebCal will be more motivated to study mathematics (increasing self-efficacy in mathematics, more positive attitude towards mathematics and a higher value of knowledge of mathematics) than students not using WebCal.
    e. Students in Collaborative Mode of Instruction sections will also be more motivated to study mathematics than students in Lecture Mode of Instruction sections.
    f. There will be a positive interaction between the impact of using WebCal and being in a Collaborative Mode of Instruction section on student motivation to study mathematics.

2. The performance of students using WebCal in Collaborative Mode of Instruction will improve if formative feedback is provided during classroom experimentation as opposed to the provision of feedback after experimentation is over.[3]

3. Comparison of the performance of students using Maple versus students not using computers at all:
   a. Students using Maple will outperform students not using Maple.
   b. Students in Collaborative Mode of Instruction sections will outperform students in Lecture Mode of Instruction sections.
   c. There will be a positive interaction between the impact of using Maple and being in a Collaborative Mode of Instruction section on student performance.
   d. Students using Maple will be more motivated to study mathematics (increasing self-efficacy in mathematics, more positive attitude towards mathematics and a higher value of knowledge of mathematics) than students not using Maple.
   e. Students in Collaborative Mode of Instruction sections will also be more motivated to study mathematics than students in Lecture Mode of Instruction sections.
   f. There will be a positive interaction between the impact of using Maple and being in a Collaborative Mode of Instruction section on student motivation to study mathematics.

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[3] Information about the experiment concerning the use of feedback in WebCal based sections, that is, concerning hypothesis 2, was published as part of a chapter in the book "Learning and teaching with technology: Principles and practices" (S. NAidu, Ed.) and delivered as a paper at the conference CATE, Rhodes, Greece 2003 and published in the conference proceedings. Thus, this information is presented in a separate chapter and so is not included in the methodology, results, discussion, etc., chapters of this report.

Methodology.

Research design

A 2x2 Factorial Design (Campbell&Stanley, 1963; Campbell&Cook, 1979; Abrami, Cholmsky & Gordon, 2001) was used in two studies to compare the effectiveness of integration of computer technology and of collaborative learning activities. The two studies refer to two different modes of integration of technology: a web-based course using CAS and a course that uses CAS alone.

Participants:

The participants in this study were 384 students enrolled in the first semester in Fall of 2001 and Fall of 2002 at Champlain College, Vanier College and Dawson College pre-university programs. Students were registered in ten different sections of Calculus I (201-NYA-05). Due to administrative problems one group of participants (21) were registered in a section of Calculus I (201-NYA-05) given in the second semester at Dawson College. The sample of students is intended to represent the population of CÉGEP students enrolled in the pre-university science program in anglophone colleges. All participants signed an informed consent form, and both the protocol that was followed in this research and the informed consent form were approved by ethics committees at Dawson College, Champlain College and Vanier college.
(see Appendix 1)

In accordance with accepted practices at Champlain College, Vanier College and Dawson College, students are quasi-randomly assigned to sections by the Registrar of each of the colleges. The particular sections of the Calculus course used were selected because the teachers agreed to participate in this experiment. Six different instructors taught these sections and each used an instructional design that they use regularly. We anticipated that there might be pretest group differences because of possible differences in student populations at each of the Colleges and across the different terms of data gathering.

Web-Cal Intervention

Introduction

Wandering the floor of the "merchandise mart" at the 1989 National Educational Computing Conference we came upon two software booths across from each other. On our right Soft Warehouse was demonstrating Derive, a CAS system that was a giant leap forward in user friendliness and capability over their earlier product muMath. On our left was an early DOS task-switching product called Black Magic. This conjunction generated instant discussion amongst ourselves about the potential for a new kind of mathematics textbook, where theorems, examples, problems, etc., would all be interactive, allowing students to change a function definition, the *x*-axis interval of interest, and watch corresponding graphical/numerical information change instantly. Naively, we bought both products intending to generate the first electronic Calculus textbook. Sadly, after months of effort we concluded that the then current computers/operating system, and we ourselves, were not up to the task. As technology rushed forward CD Calculus texts appeared, but none of them quite captured the vision that we had shared in 1989.

Over time we ourselves were influenced by the reform movement in Calculus, particularly the text of the Calculus Consortium based at Harvard, and so the goals we set for our students in Calculus changed. We came to accept the reality that a few years after taking a Calculus course the majority of students do not remember how to perform many technical manipulations. Thus, we decided to focus on concepts, which might "stick" longer, but are harder to teach/learn/test. First, we agreed that our primary goal is to create an understanding that mathematics in general is about patterns: looking for; testing the generality of; and, proving that under specified circumstances they exist. To discover this students must be given an opportunity to generate and observe many instances of function behaviour before they can begin to see patterns and generalize (Papert, 1980, 1992). In pursuit of student conceptual understanding we are committed to the Rule of Four (balance, mix and move fluidly between graphical, numerical, verbal and analytical approaches to mathematical topics). We found that many students do not generate enough examples to make underlying patterns and concepts self-evident. Their reluctance probably lies in the time consumed in doing so. We also found that even students who commit the time and generate many examples often don't see patterns. It is possible that the work of generating examples overtaxes their mental capacity, leaving no reserve capacity with which to observe patterns and concepts. Second, we want students to realize that Calculus is the mathematics of change and we have noticed that static diagrams presented in conventional texts fail to convey this notion. Third, we noticed that in our classroom practice we rarely meet the needs of individual students in terms of their different learning styles and the speed at which they learn.

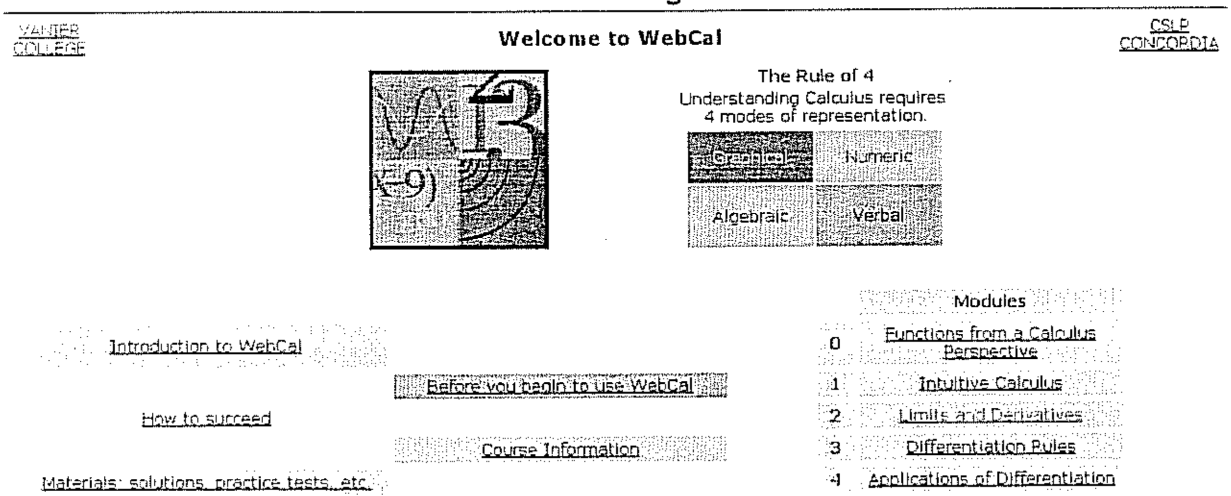
In 1998, almost a decade after our initial inspiration, we began to construct our own interactive web-based materials, WebCal, for use in the teaching of Differential Calculus. In pursuit of this project we used new technology because we felt that it would allow us to remedy difficulties faced in teaching towards our objectives while catering to our individual students' needs. Consistent with our original 1989 vision we decided to program mathematical experiments using CAS software and embed these programs within an exposition of the ideas and techniques of Differential Calculus. One purpose was to remove the burden of mechanical tasks, allowing students to generate examples rapidly and freeing student mental capacity to focus on patterns and concepts. A second purpose was to replace static diagrams with animations, helping students to more clearly understand the notion of change.

We also wished to minimize student costs. Amongst the various CAS software packages we found LiveMath (http://www.livemath.com, formerly MathView, formerly Theorist) to be unique. This software allows authors to create files containing mathematical experiments (hereafter called LiveMath inserts) that can be inserted in web pages, and all the student requires is a freely available plug-in. With the plug-in installed within a browser the student can manipulate values/formulae and see the corresponding changes in related values/formulae/graphs. Although LiveMath is less powerful than the more popular Maple and Mathematica, and because of that much more difficult to program, it is inexpensive for teachers to buy, and allows free use by students of their teachers' labour.

WebCal materials

Since virtually all of our students have computers and internet links at home (and the remaining few have access at school), we decided to place the materials on the web, allowing our students to control sequencing and pacing of learning as well as providing access to WebCal at any time.

Home Page

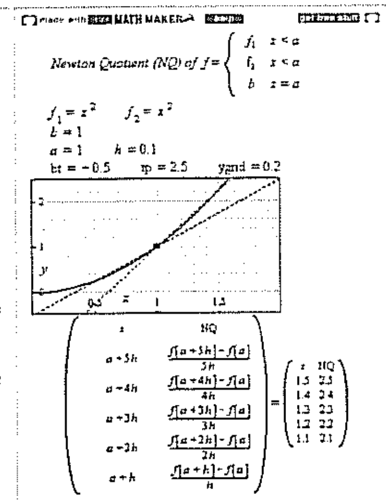


WebCal divides the content of Differential Calculus into five modules: Functions from a Calculus Perspective; Intuitive Calculus; Limits and Derivatives; Differentiation Rules; and Applications of Differentiation. Based on the recommendation of Harvard Consortium, the first two modules begin with a thorough review of functions. Students are not prepared to learn concepts of Calculus since their prior knowledge of functions is predominantly algebraic/symbolic. Functions are presented in algebraic form and their behaviour studied both graphically and numerically. Students are lead to investigate and discuss the behaviours and make verbal, algebraic, graphical and numerical predictions and observations. They also learn how complicated functions are generated from simple functions. This gives them insight into behaviours of complicated functions. The figure below shows a typical exercise from this website.

Multiple Perspectives

Exploration 1 | You can use LiveMath to compare the left-hand limit with the right hand limit for a function at x=a

The function displayed by LiveMath is a power function. Decrease the value of h and see how the secant lines (red from the right and blue from the left) approach the same tangent line. Note that you may have to change the vertical range of the graph to see the details. See the same result in the two tables which plot the values of Newton's Quotient. (Table beside graph shows right-hand values while table below shows left-hand values.) Now, use LiveMath to explore functions for which the left-hand limit is different from the right-hand limit. Set $f_2 = x^4$ and gradually decrease the value of h. The function f is continuous but the secant lines (red from the right and blue from the left) don't approach the same tangent line. In your next exploration, change b to 2 and f_2 to $x^2 + 1$ and then observe the behaviour of the two secant lines while decreasing h. You may also change f_2 to x^2 and leave $b = 2$.



The review discusses properties of families functions (power functions, polynomials, rational functions, etc.) in relation to their behaviours, *e.g.*, how they behave at the edges, their rate of growth. Students discover and develop an intuitive understanding of Calculus concepts such as limits, asymptotes, continuity, local maxima and minima, concavity etc. The review of functions and intuitive Calculus take approximately six weeks.

Intuitive Calculus Concept

Exploration | Explore the behaviour of $f(x)$ at the edges of the graph and compare it to the behaviour of $h(x)$.

The numbers lt, rt, bt and tp stand respectively for the left, right, bottom and top edges of our 'window' upon the graph; if necessary, change these to gain better perspective. You may also change both grids by changing the values of xgrid and ygrid.

The red graph is a plot of $f(x)$ with numerator $p(x)$ and denominator $q(x)$. The blue graph is a plot of $h(x)$ where m is the degree of the numerator and n is the degree of the denominator .

First change both the numerator $p(x)$ and denominator $q(x)$ of $f(x)$ by changing the degree and the coefficients. Then change m and n to create $h(x)$.

It is important to concentrate on the edge behaviour and the fact that $h(x)$ has the same trends at the edges as $f(x)$. The graphs of the two functions are certainly different in the middle part of the graph and at the edges may only run parallel to each other.

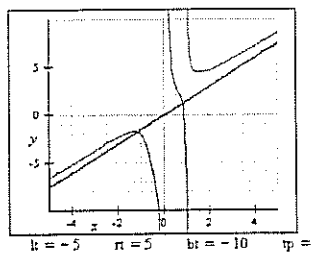
Focus your study on three cases: $m>n$; $m=n$; $m<n$. Be careful to note the effect of the sign of leading coefficients.

The edges of the graph of rational functions

$p = 3x^5+2x^4-x^2-2$

$q = 2x^4-x^2-x$

$h = \frac{3x^5}{2x^4}$



lt = −5 rt = 5 bt = −10 tp = 10
xgrid = 0.5 ygrid = 1

Each module is subdivided into sections, each section covering one topic. Each section consists of class notes (called the Lesson) and a set of solved examples that model solutions of typical problems (called Examples). In addition, built into the WebCal materials are evaluation structures and the means to attain appropriate prerequisite knowledge, both of which favour conceptual change (Pintrich et al., 1993). According to Pintrich and his colleagues (Pintrich & al., 1993), prior knowledge structure influences perception and selective attention to new information. Students may misperceive or choose to ignore data that contradicts their prior concepts. In such cases, the students' prior knowledge structure becomes a hindrance to learning. To assure that students have the appropriate prerequisite knowledge conducive to learning a new concept, WebCal provides students with prerequisite material in every section (*Prerequisites, Check-in*) as well as access to the reference module on algebra and functions, and keyword definitions so that students can refresh their memory or to learn the ideas if necessary.

Active Student Involvement

P C L E A O B F I H

reciprocal function | missing points | vertical asymptote | horizontal asymptote | peaks and valleys

Reciprocal Functions

Prerequisites

You should know the following:

1. the x-y coordinate system;
2. how to graph a function $f(x)$ given a formula;
3. power functions;

Successful learners possess appropriate meta-cognitive strategies (*e.g.*, evaluation of one's own knowledge). Consequently, to promote meta-cognition each section also includes Assignments (a list of assigned problems to practice) and Check-out (a list of competencies that students must acquire in the study of the section materials). Students use these features to assess their knowledge before proceeding to the next section.

Cognitive Awareness

P C L E A O B F I H

The Limit of a Function

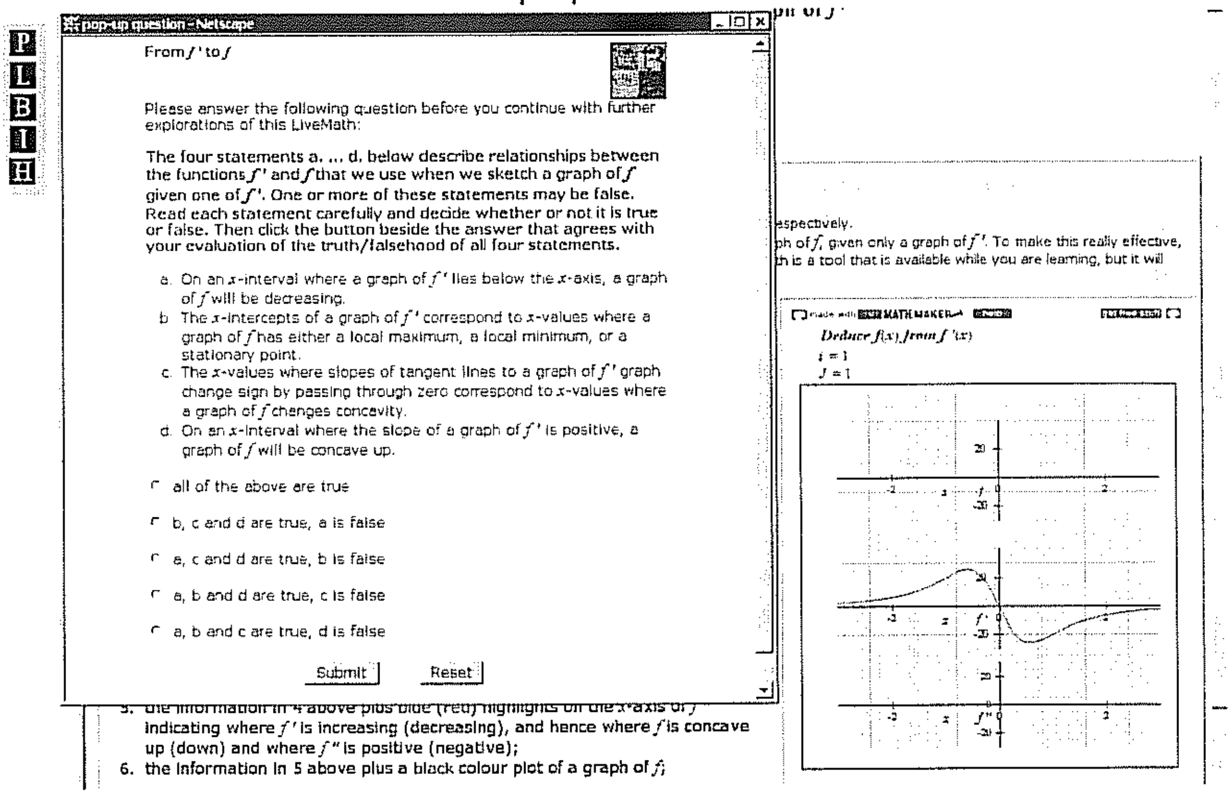
check-Out

Before you close this lesson, make sure that you can:

1. prove that a given number L is the limit of a given (simple) function $f(x)$ as x approaches some given number a;
2. prove that a given number L is not the limit of a given (simple) function $f(x)$ as x approaches some given number a;
3. given a (simple) function $f(x)$ and some particular value of x, such as a, numerically determine the limit of the function within an interval of uncertainty, and using the definition demonstrate that you indeed have found an appropriate interval of uncertainty;
4. given a function $f(x)$, its limit L as x approaches a, and a particular value for ε, determine a suitable δ;
5. given the graph of a function, determine all requested limits for the function;
6. given the limits of a function at various points, sketch a graph of the function.

In addition to Check-Out, the website includes quizzes that pop-up automatically while students use the materials. These quizzes provide immediate process oriented feedback concerning the thought process that led the student to their current observation or predictions. Moreover, when students make an erroneous prediction or observation they are given another chance.

Pop-Up Feedback

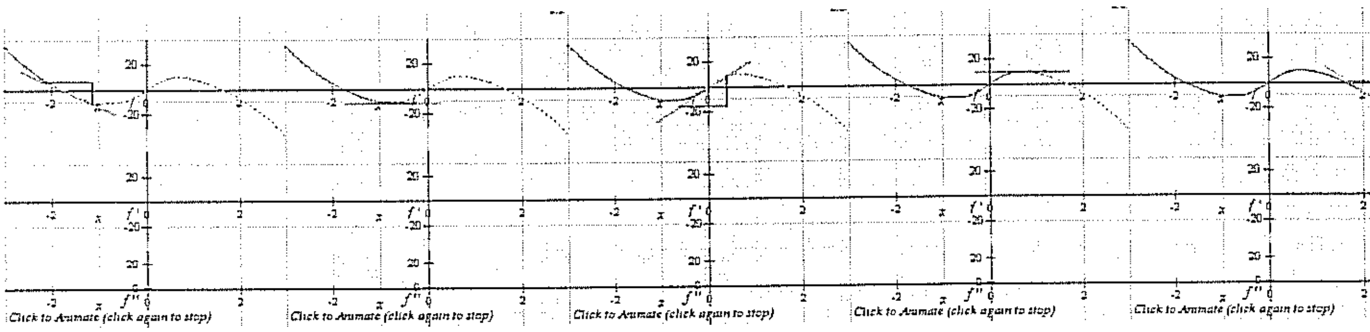


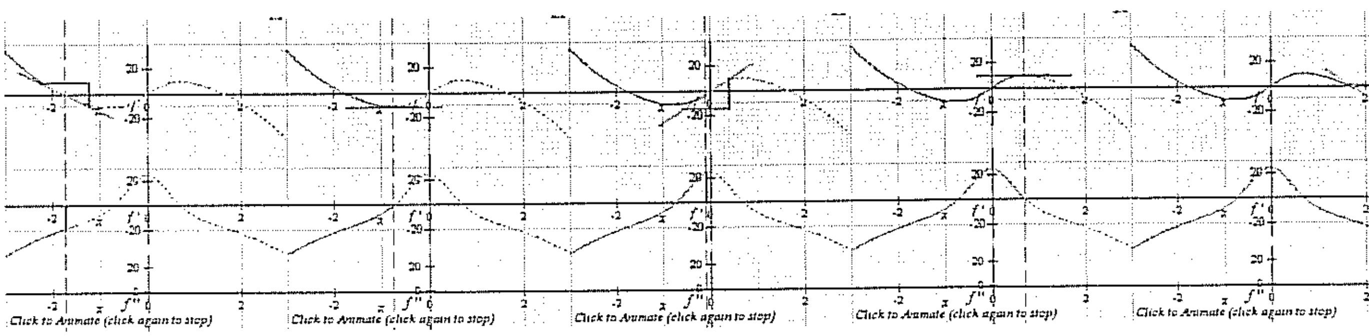
Depending upon the section, as many as three or four LiveMath inserts are embedded within the class notes, and additional LiveMath inserts may be placed within the Examples as well. LiveMath inserts are used as tools for the instructor to demonstrate complex concepts, as tools for students to experiment with functions as well as problem solving tools. Below are three examples of use of Live Math inserts. Please note that descriptions below of LiveMath inserts, as part of the WebCal materials, are impossibly inadequate. As the saying goes, a picture is worth a thousand words, so imagine the "word value" of dynamic pictures, and the impossibility of describing them in brief.

LiveMath Inserts as Demonstrations

Some concepts in Calculus are particularly difficult to grasp. For these the instructor uses LiveMath inserts as demonstrations, working at her computer, projecting the screen image for the entire class, and commenting aloud. For example, students often have trouble grasping the relationship between a function, $f(x)$, and its derivative, $f'(x)$. That is, having grasped the notion of the derivative as slope of a tangent line to $f(x)$ at a point, they fail to see the derivative of $f(x)$ as also being a function of x, and one that describes the "direction" of $f(x)$. To facilitate student understanding of such concepts we designed an animated LiveMath insert. A grid is shown with three sets of axes stacked vertically: the top for a graph of $f(x)$; the middle for a graph of $f'(x)$; and the bottom for a graph of $f''(x)$. The first animation shows a graph of $f(x)$ and a moving tangent line segment, complete with a right triangle showing "rise" and "run" as well as an analytic expression for the function $f(x)$. The second animation consists of two graphs ($f(x)$ and $f'(x)$) plotted one above the other and an analytic expression for the function $f(x)$.

Experiment to Learn





The two graphs in the second animation are plotted aligned one above another because the relationship between them is easier for students to see when they are presented in this fashion. The top graph plots $x(t)$ versus t while the bottom graph plots $v(t)$ versus t. The top graph also shows a tangent line, displayed as the hypotenuse of a right triangle, where the vertical side is the "rise" and the horizontal side is the "run". The run is set to have a constant value of 1 so that the length of the rise actually equals the slope of the tangent line. That is, the length of the rise demonstrates visually the value of the derivative (velocity) at that value of t. The LiveMath animation traces the plot of the function from left to right along the t-axis, while correspondingly moving the tangent triangle, and tracing the plot of $v(t)$ as a function of t below.

As is the case with most LiveMath inserts, this animation is complex, conveying much information simultaneously. The instructor uses it as a demonstration, pointing out animation features thereby helping students to grasp the concept. The instructor then modifies the function definition and runs the animation again. This activity allows students to draw conclusions regarding the relationships between $f'(x)$ (or $v(t)$) and $f(x)$ (or $x(t)$), while noting how the characteristics of $f(x)$ determine those of $f'(x)$ and vice versa.

LiveMath Inserts as Experiments

Newton Quotient (NQ) of $f = \begin{cases} f_1 & x < a \\ f_2 & x < a \\ b & x = a \end{cases}$

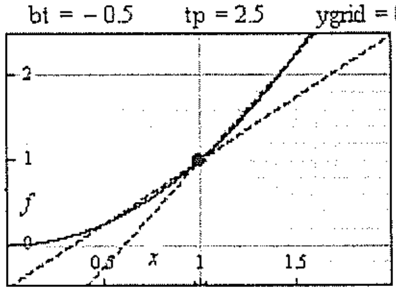
$f_1 = x^2 \qquad f_2 = x^2$

$b = 1$

$a = 1$

$h = 0.1$

bt = − 0.5 tp = 2.5 ygrid = 0.2



$$\begin{pmatrix} x & NQ \\ a+5h & \frac{f[a+5h]-f[a]}{5h} \\ a+4h & \frac{f[a+4h]-f[a]}{4h} \\ a+3h & \frac{f[a+3h]-f[a]}{3h} \\ a+2h & \frac{f[a+2h]-f[a]}{2h} \\ a+h & \frac{f[a+h]-f[a]}{h} \end{pmatrix} = \begin{pmatrix} x & NQ \\ 1.5 & 2.5 \\ 1.4 & 2.4 \\ 1.3 & 2.3 \\ 1.2 & 2.2 \\ 1.1 & 2.1 \end{pmatrix}$$

$$\begin{pmatrix} x & NQ \\ a-5h & \frac{f[a-5h]-f[a]}{[-5]h} \\ a-4h & \frac{f[a-4h]-f[a]}{[-4]h} \\ a-3h & \frac{f[a-3h]-f[a]}{[-3]h} \\ a-2h & \frac{f[a-2h]-f[a]}{[-2]h} \\ a-h & \frac{f[a-h]-f[a]}{-h} \end{pmatrix} = \begin{pmatrix} x & NQ \\ 0.5 & 1.5 \\ 0.6 & 1.6 \\ 0.7 & 1.7 \\ 0.8 & 1.8 \\ 0.9 & 1.9 \end{pmatrix}$$

LiveMath inserts are often designed to investigate a single concept and as such are well suited to be used for student experimentation. For example, to introduce the derivative as the slope of a tangent line, approximated by the slope of secant lines, students are presented with a LiveMath insert containing the following information: function definition, *e.g.*, $f(x) = x$^2; x value of interest, e.g., $x = a = 1$; initial value of h, e.g., $h = 0.1$; tables of ordered pairs (x values ranging from a-5h to a+5h, steps of size h, and y values the corresponding Newton Quotient values) presenting both symbolic expressions and numeric evaluations; and, coordinate axes displaying an appropriate piece of the graph of $f(x)$ in black, a red secant line connecting $(a, f(a))$ to $(a+h, f(a+h))$ and a blue secant line connecting $(a, f(a))$ to $(a-h, f(a-h))$.

Each pair of students is directed to modify: the value of h, controlling the accuracy of the secant approximations; the point at which the derivative is being estimated; the function. They are told to observe how numerical values of the slopes of the secant lines change as the value of h is decreased, while simultaneously the secant lines from both the left and the right merge towards a single tangent line. Thus, the student is presented with dynamically linked symbolic, numerical and graphical perspectives of this fundamental mathematical concept, and asked to verbalize with a peer. The simple effort of typing in new values (for h, x or $f(x)$) helps students to see the links between these perspectives, and hence to generate an overall understanding of the definition of the derivative as a limit of Newton's Quotient.

$$\textit{Newton Quotient (NQ) of } f = \begin{cases} f_1 & x < a \\ f_2 & x < a \\ b & x = a \end{cases}$$

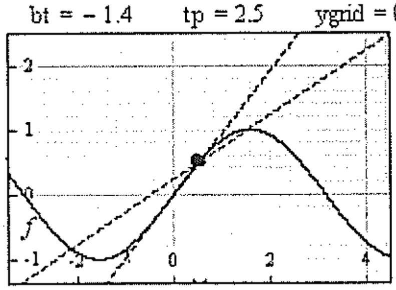
$$f_1 = \sin(x) \qquad f_2 = \sin(x)$$

$$b = \frac{1}{2}$$

$$a = \frac{\pi}{6}$$

$$h = 0.2$$

bt = − 1.4 tp = 2.5 ygrid = 0.2



$$\begin{pmatrix} x & \text{NQ} \\ a+5h & \dfrac{f[a+5h]-f[a]}{5h} \\ a+4h & \dfrac{f[a+4h]-f[a]}{4h} \\ a+3h & \dfrac{f[a+3h]-f[a]}{3h} \\ a+2h & \dfrac{f[a+2h]-f[a]}{2h} \\ a+h & \dfrac{f[a+h]-f[a]}{h} \end{pmatrix} = \begin{pmatrix} x & \text{NQ} \\ 1.5236 & 0.49889 \\ 1.3236 & 0.587 \\ 1.1236 & 0.66944 \\ 0.9236 & 0.74444 \\ 0.7236 & 0.81043 \end{pmatrix}$$

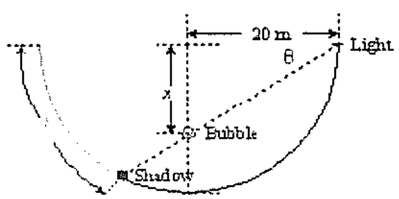
$$\begin{pmatrix} x & \text{NQ} \\ a-5h & \dfrac{f[a-5h]-f[a]}{[-5]h} \\ a-4h & \dfrac{f[a-4h]-f[a]}{[-4]h} \\ a-3h & \dfrac{f[a-3h]-f[a]}{[-3]h} \\ a-2h & \dfrac{f[a-2h]-f[a]}{[-2]h} \\ a-h & \dfrac{f[a-h]-f[a]}{-h} \end{pmatrix} = \begin{pmatrix} x & \text{NQ} \\ -0.4764 & 0.95858 \\ -0.2764 & 0.96612 \\ -0.076401 & 0.96054 \\ 0.1236 & 0.94179 \\ 0.3236 & 0.9101 \end{pmatrix}$$

Students record observations on worksheets which serve the dual purposes of providing guidance concerning what LiveMath insert manipulations might be fruitful, as well as providing a focus for concept formulation in four perspectives (see Appendix 6). The instructor assists individual groups.

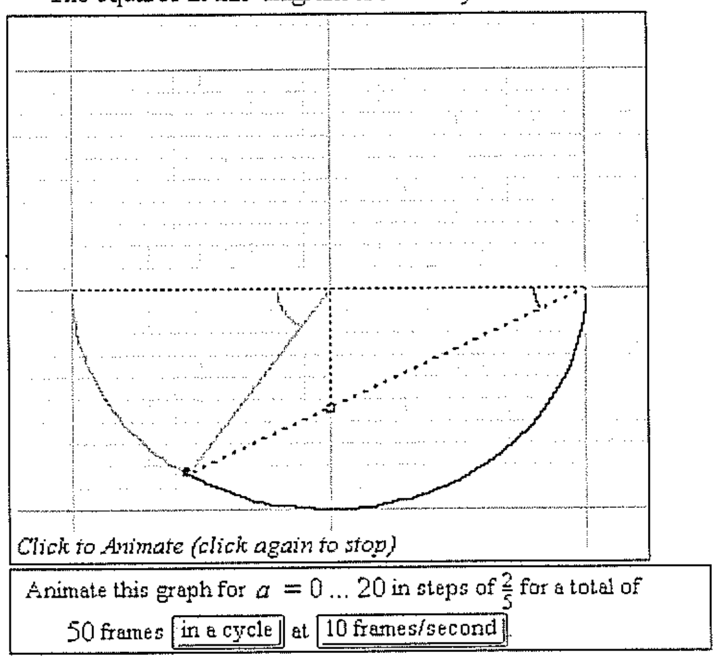
LiveMath Inserts as Problem Solving Tools

LiveMath inserts are also used as live diagrams, or simulations, to aid students in problem solving, for example in related rate word problems. Such problems are presented in textbooks with a static diagram illustrating the given situation's geometry. Unfortunately such diagrams mislead weaker students not yet capable of visualizing how changes in one (or more) variables cause changes in another (or others), or which variables will remain constant, both of which observations are the essence of related rate problems. Animated versions act as initial support enabling the weaker students to develop this type of understanding.

A hemispherical pool of radius 20 m is filled with champagne. A bubble forms on the bottom of the pool and rises up the central axis at a speed of 0.2 m/s. A light is fixed to the side of the pool at the surface and casts the shadow of the bubble on the opposite side. Champagne being what it is, the following problems have elegant solutions.



The squares in this diagram are 2 m by 2 m.



Students use these simulation tools when working on collaborative quizzes. A set of such quizzes can be found in Appendix 6.

Variables

The independent variable in this study is the instructional setting. There are six different instructional settings: Lecture WebCal; Collaborative WebCal; Lecture Maple; Collaborative Maple; Lecture non-WebCal/Maple; Collaborative non-WebCal/Maple.

There were two types of dependent variables: measures of student performance;

changes in student motivational characteristics. Student data on the measures of performance were obtained throughout the semester. Changes in motivational variables were measured as differences between responses to a questionnaire administered at the beginning of the term and again at the end of the term (*i.e.*, post-test minus pre-test).

The following variables, each of which might change over the course of a term, or might influence student reaction to a particular instructional setting, were examined: gender; prior motivation; locus of control preferences; preference for work in groups; social interaction in groups; dependence on a structured learning environment; attitudes towards computers; and, evaluation of instruction.

To assess student characteristics a 100-item questionnaire (see Appendix 2) was administered to students during the first two weeks of any course (a pre-test) and during the last week of classes (a post-test). It included scales for the following motivational variables: goal orientation; value of knowledge of mathematics; affect towards mathematics; self-efficacy. In addition, it included scales for the following other variables: gender; locus of control; preference for work in groups; social interaction in groups; dependence on a structured learning environment; attitudes towards computers; and, evaluation of Instruction. In the paragraphs below we define each of the variables and describe in more detail how it was measured.

Independent variable

Six instructional settings:

WebCal Calculus I course without collaborative learning: Class sessions were held in a computer laboratory, with pairs of students sharing each computer, and the teacher's computer wired to a projection system. The instructor lectured using WebCal as the source of instructional material. In the course of a lecture, the instructor introduced a new concept, used LiveMath (CAS) demonstrations to clarify the concept, and solved problems on the blackboard. A six-week long review of functions included intuitive discussions of the main concepts of Differential Calculus. This was followed by precise definitions of the main concepts of Differential Calculus, and then their application. At least eight quizzes were given in the course of a semester (see Appendix 6). Students were encouraged to experiment with LiveMath inserts outside the classroom. WebCal materials (lecture notes, solved examples and LiveMath inserts), as well as a regular textbook, were the references for student work outside the classroom.

Web-Cal Calculus I course with collaborative learning: Class sessions were held in a computer lab, with pairs of students sharing each computer, and the teacher's computer was wired to a projection system. Brief lectures with demonstrations preceded collaborative activities or experimentations using LiveMath (CAS) inserts. These inserts were primarily used by students in three different modes: experiments; tools during collaborative quizzes; and, tools to solve problems. A six-week long review of functions included intuitive discussions of the main concepts of Differential Calculus. This was followed by precise definitions of the main concepts of Differential Calculus and their application. At least eight collaborative quizzes were given in the course of a semester (see Appendix 6). Groups of four students discussed the problems and their solutions. To further promote collaboration, each student in a group of four submitted their work but the instructor randomly selected one paper for grading and the same grade was given to all four students. WebCal materials (lecture notes, solved examples and LiveMath inserts), as well as a regular textbook, were the reference for student work outside the classroom.

Maple (CAS) Calculus I course without collaborative learning: Class sessions were held in a multi-media classroom (*i.e.,* equipped with a video screen and a teacher computer). The teacher lectured most of the time. In the course of a lecture the teacher introduced new concepts and solved problems on the blackboard, emphasizing algebraic skills. Occasionally (15% of time), the teacher used Maple or Cabri II to demonstrate graphical properties of functions or to illustrate a new concept. These demonstrations emphasized graphical and verbal perspectives of concepts. A short review of functions, approximately two weeks, was followed by an introduction to the main concepts of Differential Calculus and their applications. Seven Maple lab sessions were held in a computer laboratory in the course of a semester (see a sample of laboratory manuals in Appendix 4). Students working individually first learned Maple syntax and a number of Maple commands. Then they used Maple to solve problems and to explore graphs. Each student was required to submit a written report on each Maple lab. The textbook was the sole reference for student work outside the classroom.

Maple (CAS) Calculus I course with collaborative learning: Class sessions were held in a multi-media classroom (*i.e.,* equipped with a video screen and a teacher computer). The instruction given in the multi-media classroom was the same as in the setting described above. Similarly, seven Maple lab sessions were held in a computer laboratory but in this setting the students worked on the labs in groups of four. Each group learned Maple syntax and commands, and then students collaborated on tasks set before them in the laboratory manual (see Appendix 4). To further promote collaboration, each student in a group of four submitted their work but the instructor randomly selected one paper for grading and the same grade was given to all four students. The textbook was the sole reference for student work outside the classroom.

Traditional Calculus I course without collaborative learning. Class sessions were held in a regular classroom. The teacher lectured most of the time. In the course of a lecture, the teacher introduced new concepts and solved problems on the blackboard. A two-week review of functions was followed by an introduction to the main concepts of Differential Calculus and their application. The textbook was the sole reference for student work outside the classroom.

Traditional Calculus I course with collaborative learning. Class sessions were held in a regular classroom. The teacher lectured most of the time. In the course of a lecture, the teacher introduced new concepts and solved problems on the blackboard. A two-week review of functions was followed by the introduction of the main concepts of Calculus and their application. Eight collaborative sessions were held during the semester. To further promote collaboration, each student in a group of four submitted their work but the instructor randomly selected one paper for grading and the same grade was given to all four students. The textbook was the sole reference for student work outside the classroom.

Dependent Variables

Measures of Achievement:

Achievement was assessed by measuring student performance on six different problems that all the instructors inserted into their regular term tests or final examinations.[4] These problems span all major topics of Calculus I. Two problems on limits include one problem which mostly tests students ability to compute the limits using the rules and another problem which tests student conceptual understanding of the concept of limits at infinity. The function in the second problem on limits is presented graphically and students are asked to determine the limits from their examination of the given graph of the function. Two other problems that test students' ability to compute the derivatives were used: one requiring use of the definition of derivative (Newton Quotient); another requiring use of the rules of differentiation (*e.g.*, product rule, chain rule, etc.). The last two problems test students' ability to synthesize the concepts of limit and derivative and use them to deduce features in a graph of a given function (a polynomial function and a rational function). All six problems are typical Calculus I problems that could be found on any traditional examination. We have developed a coding schema for each of the six problems (see Appendix 3). The coding schema were tested and improved until inter-coder reliability exceeded 85%. Then two coders coded all papers.

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[4]Note that the researchers and the instructors jointly developed a set of fourteen problems that was intended to be used to measure students' performance. Unfortunately, through a series of mishaps over the period of three semesters we only have six problems that were given to all students under the sufficiently similar conditions. For example, one of the problems was eliminated because an instructor included it as a bonus question rather than a compulsory question; another problem was eliminated because two of the instructors gave additional instructions to the students while the examination was in progress. This type of difficulty arises primarily because most CÉGEP teachers are not used to being involved in educational research projects. As an unfortunate consequence of this difficulty, the final coded set of six problems is missing representatives of two important applications of derivatives, *i.e.*, related rate and optimization problems.

The coding schema provided us with a record of all the steps that each student took while solving a problem. The schema also included a count of errors and type of errors that each student made. A scoring schema (see Appendix 3) was then developed to evaluate students' arithmetic/algebraic skills (a count of algebraic errors) and the ability to correctly use symbolic notation (a count of errors in using mathematical symbols correctly: e.g.,student writing $\lim_{x\to 2} = 3$ instead of $\lim_{x\to 2}(x+1) = 3$ was counted as an error in using symbolic notation). Students' understanding of graphs was evaluated by coding how students used computed data about properties of a function (e.g.; relationship between the sign of the second derivative and concavity of the curve) to draw a graph. Students' ability to correctly use algorithms (a student may use a correct algorithm but arrive at a wrong answer because of an algebraic error; inversely a student may use an algorithm incorrectly but still arrive at a correct answer) was also evaluated. In addition, students' conceptual understanding was assessed by counting errors made in reasoning. Finally, students' ability to solve problems and arrive at a correct answer without making an error was also assessed by counting correctly answered problems.

Traditionally, instructors' grading of a solution of a problem is a composite of obtaining a correct answer, extent to which a correct algorithm was used and absence or presence of algebraic and symbolic errors. The instructors often differ in how much weight they give to each of the above elements of the solution. Some instructors may only give a full grade when the answer is correct and assign zero grade when any error is made. Other instructors give partial grades by taking into account the type of errors and the seriousness of the errors made. For example, a student may still get a high partial score if only algebraic errors are made. Obviously, there is some arbitrariness and subjectivity (bias) in traditional grading. To retain objectivity, all student work was photocopied for coding prior to teacher grading, and we avoided using any composite grades for any of the problems in this study.

Motivational variables

Goal Orientation: A mastery learning orientation results in the most adaptive responses, such as an increased effort to solve a problem or more perseverance when confronted with a difficult situation (Roedel, Schraw, & Plake, 1994). Conversely, a performance goal orientation is likely to be reflected in maladaptive responses, and is characterized by a focus on outcomes and a desire to avoid negative feedback. A 9-item goal orientation scale was adapted from a scale developed by Zweig (2001). It assesses students' desire to understand concepts of mathematics rather than to just get a good grade. A low score on this scale indicates that a student is more mastery oriented than performance oriented. To illustrate this point a typical item is given below:

When I don't understand ideas presented in mathematics courses,
a) it doesn't bother me at all; I only care about my grades.
b) it bothers me a little but if my grades are already good I will not try to fix it

c) it bothers me a lot but if my grades are already good I will not try to fix it.
d) it bothers me a lot. Even if my grades are already good I will try to fix it.
e) it bothers me a lot. Even if my grades are already good I will not stop until I have fixed it.

A mastery oriented student is likely to respond e) while a performance oriented student can be expected to choose a) because he is primarily concerned with grades.

Value of knowledge of mathematics: When students are convinced that the knowledge of a subject is important for them they are more likely to expend the effort required to grasp difficult concepts and to persevere when faced with difficulties. A 7-item value of knowledge of mathematics scale was adapted from Opinions sur les mathématiques (Lafortune, 1992; Collette, 1976). It assesses students' perception of usefulness of the knowledge of mathematics. A low score on this scale indicates that a student perceives mathematics to be useful. To illustrate this point a typical item is given below:

Mastery of basic math concepts is a prerequisite for my future studies.
a) strongly agree
b) agree
c) neither agree nor disagree
d) disagree
e) strongly agree

Affect towards mathematics: Students who have a positive emotional response (enjoy, fun, comfort) to learning a subject are more likely to overcome obstacles than students who have a negative emotional response (hate, dislike, frustration). A 10-item affect towards mathematics scale was adapted from Opinions sur les mathématiques (Lafortune, 1992; Collette, 1976). It assesses students' feelings about mathematics. Students who score low on this scale have positive emotions about mathematics. To illustrate this point here is a typical item:

I have fun solving hard problems in math.
a) strongly agree
b) agree
c) neither agree nor disagree
d) disagree
e) strongly agree

Self-efficacy: Students who perceive themselves as capable of learning mathematics and capable of solving problems in mathematics are likely to expend more effort and persevere in their studies of mathematics than students who perceive themselves as failures (Hall & Ponton, 2002; Bandura, 1997). This 8-item scale has been adapted from *Motivated Strategies for Learning Questionnaire* (Pintrich. et. al., 1991). This measure assesses students' self-perception of competence in mathematics. Students who score low on this scale are students who are confident about their success. To illustrate this point here is a typical item:

I can succeed in math.

a) always
b) usually
c) sometimes
d) rarely
e) never

Beliefs about knowledge of mathematics: It is believed that epistemological beliefs guide student actions. The Epistemological Beliefs Questionnaire (EBQ) was initially developed by Schommer (1990) to assess students' beliefs about knowledge and learning in general. The original version of EBQ was subsequently validated by Schommer et al. (1992). Qian and Alvermann (1995) refined and validated a slimmed down version of EBQ by eliminating roughly half of the items, bringing it down from four factors to two. Elby (2000) and Hammer and Elby (2000, 2002) argue that these beliefs may not be concepts that students develop consciously. The Schommer instrument is very general. One cannot assume that assessment of students' beliefs about knowledge in general will also assess their beliefs about mathematics (Hofer & Pintrich,1997). Consequently we have developed, in collaboration with Dr. Ilona Jerabek of psychtest.com, a scale that assesses college students' beliefs about: mathematics; learning mathematics; and, talent for mathematics. Students who have a low score on beliefs about mathematics are students who believe that understanding mathematics implies an understanding of ideas rather than the ability to carry out problem-solving procedures. To illustrate this point a typical item is given below:

Math is

a) all about understanding general ideas.
b) mostly about understanding general ideas.
c) mostly about carrying out procedures step-by-step.
d) all about carrying out procedures step-by-step.

Students who have a low score on beliefs about learning mathematics are students who believe that mathematics is learned slowly through making and correcting mistakes. To illustrate this point a typical item is given below:

Making several unsuccessful attempts when solving math problems

a) is perfectly natural.
b) is relatively normal.
c) indicates a potential problem with student's ability to learn math.
d) indicates that a student has a problem when it comes to math.
e) a clear sign of a student who is bad in math.

Students who have a low score on beliefs about a talent for mathematics are students who believe that effort is more important for learning mathematics than having a special talent. To illustrate this point a typical item is given below:

Knowledge of mathematics

a) depends entirely on the amount of effort one puts into learning it.
b) depends mostly on the effort one puts into learning it.
c) depends equally on effort and a talent for mathematics.
d) depends mostly on one's talent for mathematics.
e) depends entirely on one's talent for mathematics.

Preference for work in groups: Students who prefer to work with others are more likely to succeed in collaborative instructional settings than are students who prefer to work alone. This 7-item scale was elaborated in collaboration with Dr. Ilona Jerabek of psychtest.com. This scale assesses students' preference for problem-solving in a group, rather than in an individual setting. A low score on this scale indicates a high preference for group work. To illustrate this point a typical item is given below:

It is useful to work on math assignments in a group because we can help each other.

a) strongly agree
b) agree
c) neither agree nor disagree
d) disagree
e) strongly agree

Social interaction in groups: Students who do not feel self-confident in a group-setting are unlikely to succeed in a collaborative-instructional setting. This 3-item scale was elaborated in collaboration with Dr. Ilona Jerabek of psychtest.com. This scale assesses students' self-confidence in a group. A low score on this scale indicates high self-confidence. To illustrate this point a typical item is given below:

In a typical group setting, I feel left out.

a) always
b) usually
c) sometimes
d) rarely
e) never

Dependence on learning environment : Students who prefer to learn in a very structured environment in which tasks and procedures are given step-by-step will feel uncomfortable in an environment in which tasks demand that they creatively explore and formulate their own conclusions. Consequently, such students are not likely to succeed in the latter type of environment. This 6-item scale was elaborated in collaboration with Dr. Ilona Jerabek of psychtest.com. This scale assesses students' preference for a well structured learning environment. A low score on this scale indicates high self-confidence. To illustrate this point a typical item is given below:

I get anxious when I don't get step-by-step instructions on how to accomplish a task.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

Attitudes towards computers: Students who are comfortable learning mathematics using computers are likely to be motivated to learn in environments in which computers are used extensively. A 10-item scale was elaborated in collaboration with Dr. Ilona Jerabek of psychtest.com. This scale assesses students' preference for using computers to learn mathematics. A low score on this scale indicates a high preference for using computers. To illustrate this point a typical item is given below:

Using computers to learn math is a waste of time.
a) strongly agree
b) agree
c) neither agree nor disagree
d) disagree
e) strongly agree

Locus of control: Students who believe that internal forces control their life outcomes are likely to feel comfortable in environments that give them an opportunity to explore on their own and to formulate their own conclusions. An 8-item scale that measures locus of control as a general trait was developed by Dr. Ilona Jerabek and is available at psychtest.com. A low score on this scale indicates a high internal locus of control. To illustrate this point a typical item is given below:

Being at the right place at the right time is essential for getting what you want in life.
a) always
b) usually
c) sometimes
d) rarely
e) never

Coping skills: Students who have highly developed coping skills are much more likely to adapt to novel environments than students who lack such skills. Consequently, student who possess these skills are more likely to succeed in mathematics courses that integrate the use of technology. A 10-item scale that measures coping skills as general life skills was developed by Dr. Ilona Jerabek and is available at psychtest.com. A low score on this scale indicates high coping skills. To illustrate this point a typical item is given below:

When the situation changes, I adjust my plans.
- a) strongly agree
- b) agree
- c) neither agree nor disagree
- d) disagree
- e) strongly agree

Evaluation of Instruction: Since six different teachers implemented different instructional settings it is legitimate to ask whether anticipated differences in student outcomes are primarily caused by differences in instructional settings or if it is possible that different outcomes could be attributed to differences in teaching styles. To minimize the impact of teacher styles we have included three particular questions in the post-questionnaire, and we used ANCOVA to correct for differences in teaching styles. In considering evaluation of instruction, only students who indicated that they attended 80% or more of classes were included. The three questions are:

1. When you compare the workload in this Calculus course to the workload in your other science courses, do you consider it to be
   - a) very heavy?
   - b) heavy?
   - c) average?
   - d) lighter?
   - e) very light?

2. In this course I attended
   - a) more than 90% of the classes.
   - b) over 80% of the classes.
   - c) over 70% of the classes.
   - d) more than half of the classes.
   - e) less than half of the classes.

3. The instruction in this course was
   - a) very good.
   - b) good.
   - c) satisfactory.
   - d) fair.
   - e) unsatisfactory

Results

Reliability of scales

Some of the scales were adapted from previously validated instruments. The validation process may not apply when items are modified to refer to mathematics alone and not science in general. In some cases the wording was also slightly changed and thus, we decided to re-validate them. Other scales were developed specifically for this study and needed to be validated. We have used α-Cronbach calculation to assess reliability. A high α-Cronbach value indicates high correlation between items (see the mean correlation between items in the last column of Table 1 below). We also list values of mean item variances although we did not use reliability tests that assume equal variances. We only considered cases with no missing values. Hence the number of cases in each scale is different (the second column in the Table 1). The number of items in each scale is listed in column 3 and the α-Cronbach value of the scale is shown in column 4. The results indicate that the scales goal orientation, value of knowledge of mathematics, affect towards mathematics, self-efficacy, preference for group works, attitudes towards computers and coping skills are internally consistent. Both dependence on learning environment and locus of control are much less internally consistent, as the average of mean item correlations indicates. The social interaction in groups scale has only three items which probably accounts for its low α-Cronbach value, although mean item correlation is not too low.

Note that the beliefs about knowledge scale is not presented in the results. Our reliability analysis revealed that items in this scale do not correlate. The correlation matrix showed very small correlations. In addition, some correlations were positive and others were negative. We subsequently attempted to determine factors within this scale using factor analysis. Only three factors could be found using this procedure, but the correlations were weak. We used reliability analysis again to test the reliability of those factors but the α-Cronbach values were still unacceptably low. Furthermore, in our expert opinion the set of items in any one of those three factors did not seem to address a single particular aspect of beliefs about knowledge of mathematics. We note that one possible reason is that beliefs about knowledge are neither stable nor consistent (Elby, 2001). Consequently, we cannot expect that items addressing the same idea about knowledge necessarily correlate. We intend to use structured interviews in the future in an attempt to validate such items. Only then will we be able to examine whether students' epistemological beliefs impact on their learning in different learning environments.

Table 1.

Variable Name	Number of Cases	Number of Items	α-Cronbach	Mean Item variances	Mean Item Correlations
Goal Orientation	368	9	0.7878	0.7381	0.2921
Value of knowledge of mathematics	373	7	0.7038	0.8491	0.2534
Affect towards mathematics	368	7	0.8668	0.8627	0.4485
Self-efficacy	377	8	0.8361	0.7190	0.3893
Preference for work in groups	374	7	0.7458	0.9367	0.2953
Social interaction in groups	375	3	0.5608	0.8915	0.2985
Dependence on learning environment	377	6	0.6005	1.0620	0.2003
Attitudes towards computers	373	8	0.8833	1.2384	0.4863
Locus of control	365	8	0.6191	1.0780	0.1689
Coping skills	372	10	0.7738	0.7799	0.2549

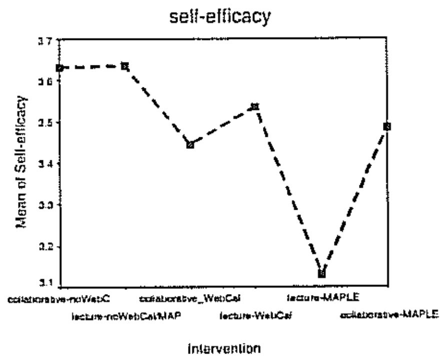
A comparison of student characteristics.

Participants in this study were taught Calculus I using six different instructional settings. As mentioned above, students' enrollment in Calculus sections was not random. Consequently, we needed to test whether means of student characteristics differed across the six different instructional settings. We used one-way ANOVA to test the hypothesis that the means are equal in all six settings. Since the number of participants is not the same in each of the conditions we tested whether the samples had equal variances (Levene's test of homogeneity).

We found that there were no significant differences between the means on all but five of the characteristics: self-efficacy; preference for work in groups; social interaction in groups; attitude towards computers; coping skills. The results are presented both in table format (Table 2 below) and as a plot below the table. N indicates the number of students in each of the six conditions.

Table 2. Self-efficacy

	N	Mean	Std. Dev.	F	Sig.
Collaborative; non-WebCal/Maple	37	3.632	.557	3.299	0.006
Lecture; non-WebCal/Maple	98	3.635	.576		
Collaborative; WebCal	73	3.445	.562		
Lecture; WebCal	70	3.535	.568		
Collaborative; Maple	20	3.131	.555		
Lecture; Maple	33	3.485	.531		
Total	331	3.526	.574		

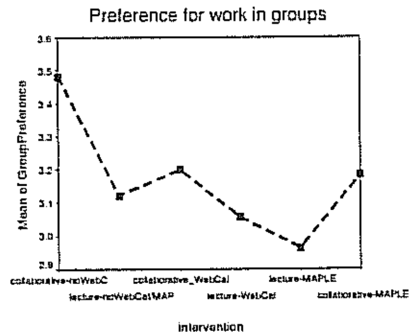


Students also significantly differ ($p = 0.006$) in self-efficacy. The plot above shows that students Lecture sections using Maple have higher self-efficacy feelings than students in other sections.

Students also significantly differ ($p = 0.006$) in their preference for working in groups when studying or solving problems in mathematics. N indicates the number of students in each of the six conditions (Table 3 below). The means and standard deviations for each group are also given.

Table 3. Preference for work in groups

	N	Mean	Std. Dev.	F	Sig.
Collaborative; non-WebCal/Maple	37	3.483	.709	3.070	0.010
Lecture; non-WebCal/Maple	99	3.122	.583		
Collaborative; WebCal	73	3.199	.529		
Lecture; WebCal	70	3.057	.598		
Lecture; Maple	33	3.183	.648		
Collaborative; Maple	20	2.964	.727		
Total	332	3.162	.615		

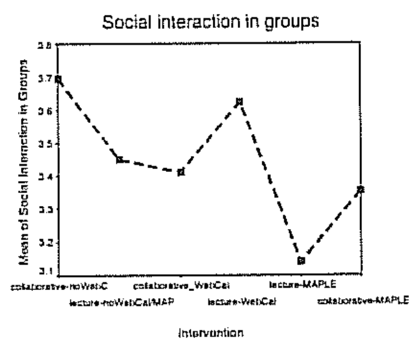


The plot above shows that students in non-WebCal/Maple Collaborative sections preferred to work individually when studying and solving problems in mathematics.

Similarly, students also significantly differ ($p = 0.023$) in their social interaction in groups when studying or solving problems in mathematics. N indicates the number of students in each of the six conditions (Table 4 below). The means and standard deviations for each group are also given.

Table 4. Social Interaction in groups

	N	Mean	Std. Dev	F	Sig.
Collaborative; non-WebCal/Maple	36	3.694	.723	2.654	0.023
Lecture; non-WebCal/Maple	98	3.449	.746		
Collaborative; WebCal	73	3.411	.620		
Lecture; WebCal	69	3.623	.605		
Lecture; Maple	19	3.140	.826		
Collaborative; Maple	33	3.354	.661		
Total	328	3.477	.694		

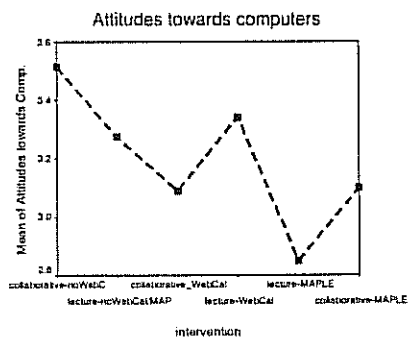


The plot above shows that students in the Maple Lecture section are more confident in groups.

In addition, students also significantly differ ($p = 0.031$) in their attitudes towards computers. N indicates the number of students in each of the six conditions (Table 5.). The means and standard deviations for each group are also given.

Table 5. Attitudes towards computers.

	N	Mean	Std. Dev.	F	Sig.
Collaborative; non-WebCal/Maple	37	3.514	.715	2.501	0.031
Lecture; non-WebCal/Maple	99	3.274	.827		
Collaborative; WebCal	73	3.090	.984		
Lecture; WebCal	70	3.342	.870		
Lecture; Maple	20	2.850	.834		
Collaborative; Maple	33	3.100	.627		
Total	332	3.232	.855		

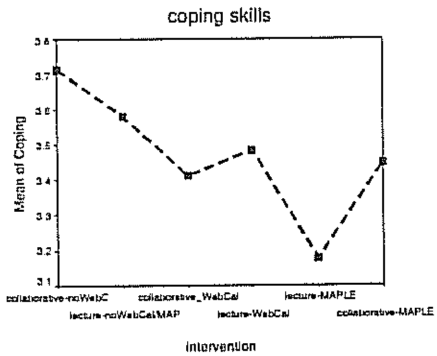


The plot above shows that students in the Maple Lecture section have a more positive attitude towards computers than students in non-WebCal/Maple Collaborative sections.

Lastly, students also significantly differ ($p = 0.002$) in their coping skills. N indicates the number of students in each of the six conditions (Table 6.). The means and standard deviations for each group are also given.

Table 6. Coping skills

	N	Mean	Std. Dev.	F	Sig.
Collaborative; non-WebCal/Maple	37	3.715	.508	3.978	.002
Lecture; non-WebCal/Maple	98	3.58	.499		
Collaborative; WebCal	73	3.411	.537		
Lecture; WebCal	70	3.484	.505		
Lecture; Maple	20	3.178	.557		
Collaborative; Maple	33	3.448	.413		
Total	331	3.500	.517		



The plot above shows that students in the Maple Lecture section have a more coping skills than students non-WebCal/Maple Collaborative sections.

The effectiveness of WebCal usage.

To examine the effectiveness of WebCal usage, we compared WebCal Lecture and Collaborative sections with non-WebCal Lecture and Collaborative sections, using a 2 x 2 factorial design. (Note that the term non-WebCal sections does **not** include sections that used Maple.) Given that four different instructors taught these sections we had to test for the impact of differences between the instructors. A one-way ANOVA analysis was conducted where the independent variable was the instructor and the dependent variable was evaluation of instruction. Recall that this latter variable depends upon students' class attendance and their global assessment of instruction. We found no significant differences amongst the four instructors ($F = 1.734$, $p = 0.161$).

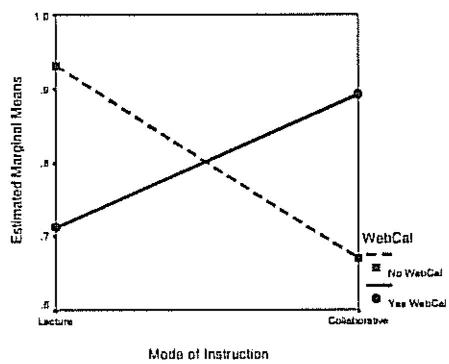
Given this result, we used two-way ANOVA to study whether there are significant differences between the means in various settings. In each of the results reported below we tested whether the samples had equal variances (Levene's test of homogeneity). The factors were: Mode of Instruction, with two levels (Lecture and Collaborative) and WebCal Usage, with two levels (non-WebCal and WebCal). Below we present only those results of the analysis that were statistically significant. First, we will focus on the impact on measures of achievement, and then we will show how WebCal Usage and Mode of Instruction impacted student motivation to study mathematics.

Table 7 below provides descriptive statistics, mean, standard deviation (St. Dev.) and sample size in each cell, as well as totals for each (row, column) pair of the 2 x 2 factorial design, for the outcome variable arithmetic and algebraic skills. The last three columns of the table show the two main effects and any interaction between Mode of Instruction and WebCal Usage. (The tables below have a similar structure.) As one can see in Table 7, there was no significant main effect, but there was a significant interaction between Mode of Instruction and WebCal Usage ($p = 0.009$).

Table 7. Arithmetic and Algebraic Skills

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-WebCal	0.93	0.801	67	Mode of Instruction	0.233	0.630
	WebCal	0.71	0.576	64	WebCal Usage	0.000	0.986
	Total	0.82	0.706	131	Mode of Instruction*WebCal Usage	6.895	0.009
Collaborative	non-WebCal	0.67	0.377	35			
	WebCal	0.89	0.563	72			
	Total	0.82	0.518	107			
Total	non-WebCal	0.84	0.695	102			
	WebCal	0.81	0.574	136			
	Total	0.82	0.628	238			

a R Squared = 0.029 (Adjusted R Squared = 0.017)



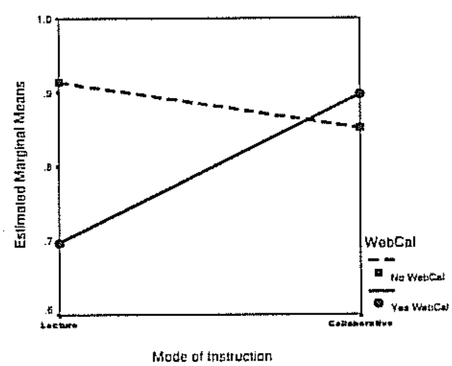
The graph above shows that while WebCal students outperformed non-WebCal in Lecture sections the non-WebCal students outperformed WebCal students in Collaborative sections.

Table 8 below provides results for the outcome variable, use of symbolic language. The last three columns show that there was no significant difference between students in different instructional settings. However, the interaction between Mode of Instruction and WebCal Usage on the outcome variable concerning how students use symbolic language is significant ($p < 0.080$).

Table 8. Use of Symbolic Language

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-WebCal	0.91	0.518	64	Mode of Instruction	0.861	0.354
	WebCal	0.70	0.438	66	WebCal Usage	1.332	0.250
	Total	0.80	0.489	130	Mode of Instruction*WebCal Usage	3.098	0.080
Collaborative	non-WebCal	0.85	0.620	37			
	WebCal	0.90	0.640	72			
	Total	0.88	0.630	109			
Total	non-WebCal	0.89	0.559	101			
	WebCal	0.80	0.559	138			
	Total	0.84	0.558	239			

a R Squared = 0.026 (Adjusted R Squared = 0.014)



The graph above shows that while WebCal students outperformed non-WebCal in Lecture sections, non-WebCal students outperformed WebCal students in Collaborative sections.

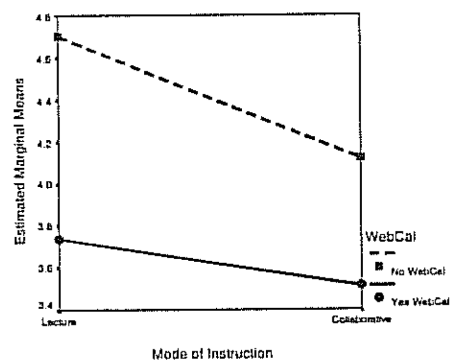
Table 9 below provides results for the outcome variable, understanding of graphs. From the table we see that students' understanding of graphs differs across sections. The last three columns show two significant main effects: both Mode of Instruction and WebCal Usage impact significantly on students' performance on understanding of graphs ($p = 0.034$ and $p < 0.001$ respectively).

Table 9. Understanding of Graphs

Mode of Instruction		Descriptive		
		Mean	St. Dev.	N
Lecture	non-WebCal	4.70	0.680	62
	WebCal	3.74	1.439	65
	Total	4.21	1.229	127
Collaborative	non-WebCal	4.12	1.556	33
	WebCal	3.51	1.649	72
	Total	3.70	1.637	105
Total	non-WebCal	4.50	1.096	95
	WebCal	3.62	1.551	137
	Total	3.98	1.447	232

Tests between subjects effects		
Source	F	Sig.
Mode of Instruction	4.525	0.034
WebCal Usage	17.223	0.000
Mode of Instruction*WebCal Usage	0.884	0.348

a R Squared = 0.108 (Adjusted R Squared = 0.097)



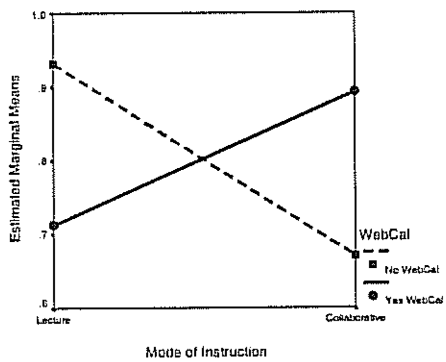
The graph above indicates that students in Collaborative sections outperformed students in Lecture sections. It also shows that WebCal students outperformed non-WebCal students.

Table 10 below provides results for the outcome variable, use of algorithms. From the table we can see that students' use of algorithms differs across the sections. The last three columns demonstrate that there was only one effect of significance ($p = 0.009$), the interaction effect between Mode of Instruction and WebCal Usage.

Table 10. Use of Algorithm

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-WebCal	0.93	0.801	67	Mode of Instruction	0.233	0.630
	WebCal	0.71	0.576	64	WebCal Usage	0.000	0.986
	Total	0.82	0.706	131	Mode of Instruction*WebCal Usage	6.895	0.009
Collaborative	non-WebCal	0.67	0.377	35			
	WebCal	0.89	0.563	72			
	Total	0.82	0.518	107			
Total	non-WebCal	0.84	0.695	102			
	WebCal	0.81	0.574	136			
	Total	0.82	0.628	238			

a R Squared = 0.029 (Adjusted R Squared = 0.017)



The graph above indicates that in Lecture sections, WebCal students performed better in correctly using algorithms than non-WebCal students. It also shows that the situation reversed itself in Collaborative sections, where non-WebCal students demonstrated a higher performance in using algorithms.

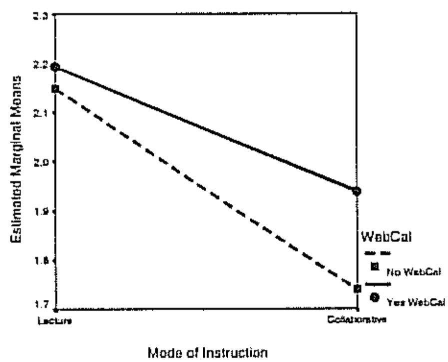
There was neither any main effect nor an interaction effect for the outcome variable understanding of concepts.

Table 11 below provides results for the outcome variable, correct answer. From the table we see that students' ability to arrive at the correct answer differs across sections. The last three columns show one significant main effect: Mode of Instruction impacts significantly on students' ability to arrive at the correct answer ($p < 0.001$).

Table 11. Correct Answer

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-WebCal	2.15	0.711	66	Mode of Instruction	6.99	0.009
	WebCal	2.19	1.373	69	WebCal Usage	0.946	0.332
	Total	2.17	1.097	135	Mode of Instruction*WebCal Usage	0.386	0.535
Collaborative	non-WebCal	1.74	0.740	36			
	WebCal	1.94	0.668	73			
	Total	1.87	0.696	109			
Total	non-WebCal	2.00	0.744	102			
	WebCal	2.06	1.074	142			
	Total	2.04	0.949	244			

a R Squared = 0.029 (Adjusted R Squared = 0.017)



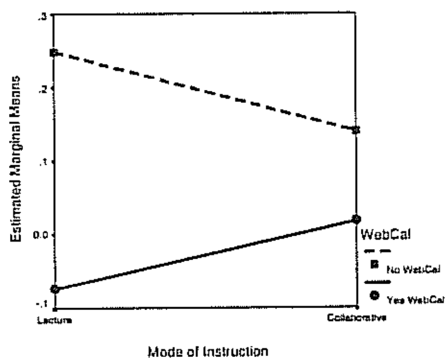
The graph above indicates that students in Collaborative sections outperformed students in Lecture sections in their ability to solve problems without making errors.

Next we examined the differential effects of instructional settings on students' motivation to study mathematics. There were no significant differences in change of any of the motivational variables among students in different instructional settings with the exception of one significant main effect ($p = 0.002$) on the change in dependence on a structured environment. Table 12 below shows this result.

Table 12. The Change in Dependence/Independence

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-WebCal	0.25	0.536	54	Mode of Instruction	0.015	0.903
	WebCal	-0.07	0.556	63	WebCal Usage	9.834	0.002
	Total	0.07	0.568	117	Mode of Instruction*WebCal Usage	2.000	0.159
Collaborative	non-WebCal	0.14	0.435	31			
	WebCal	0.02	0.437	70			
	Total	0.06	0.438	101			
Total	non-WebCal	0.21	0.502	85			
	WebCal	-0.03	0.497	133			
	Total	0.07	0.511	218			

a R Squared = 0.060 (Adjusted R Squared = 0.046)



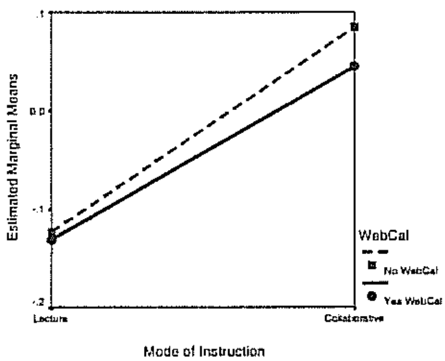
The graph above indicates that WebCal students became less dependent on a structured environment during the course of the semester in comparison with non-WebCal students who became more dependent.

Although the following results are unrelated to our hypotheses, we believe that they are interesting to note. We found a significant main effect on changes in students' attitudes towards computers as a learning tool ($p = 0.047$). (See Table 13 below.)

Table 13. The Changes in Students' attitudes towards Computers

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-WebCal	-0.12	0.515	55	Mode of Instruction	3.989	0.047
	WebCal	-0.13	0.872	63	WebCal Usage	0.061	0.805
	Total	-0.13	0.724	118	Mode of Instruction*WebCal Usage	0.027	0.870
Collaborative	non-WebCal	0.08	0.425	31			
	WebCal	0.05	0.680	70			
	Total	0.06	0.611	101			
Total	non-WebCal	-0.05	0.492	86			
	WebCal	-0.04	0.779	133			
	Total	-0.04	0.679	219			

a R Squared = 0.019 (Adjusted R Squared = 0.005)



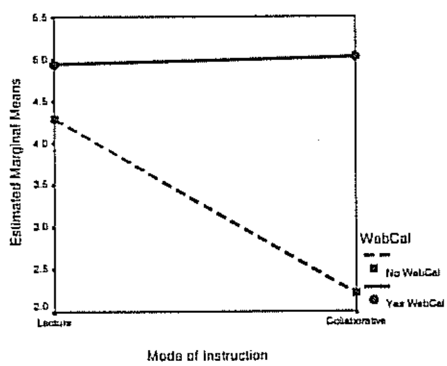
We can see from the graph that while students in Lecture sections became more positive about using computers in Lecture sections students in Collaborative sections became more negative.

The post-questionnaire asked students to report the number of hours they studied mathematics (Study Hours), how they perceived the course workload (Workload) and whether they were confused when leaving a class room (Class Feeling). Although these results were not significant, we include them in our discussion because they may be related to changes in students' attitudes towards using computers as learning tools. (See Tables 14, 15 and 16 below.)

Table 14. Study Hours

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-WebCal	4.282	3.6334	55	Mode of Instruction	2.973	0.086
	WebCal	4.935	5.4151	62	WebCal Usage	9.100	0.003
	Total	4.628	4.6554	117	Mode of Instruction*WebCal Usage	3.523	0.062
Collaborative	non-WebCal	2.217	1.4544	30			
	WebCal	5.023	3.3836	65			
	Total	4.137	3.1886	95			
Total	non-WebCal	3.553	3.1941	85			
	WebCal	4.980	4.4736	127			
	Total	4.408	4.0626	212			

a R Squared = 0.054 (Adjusted R Squared = 0.040)



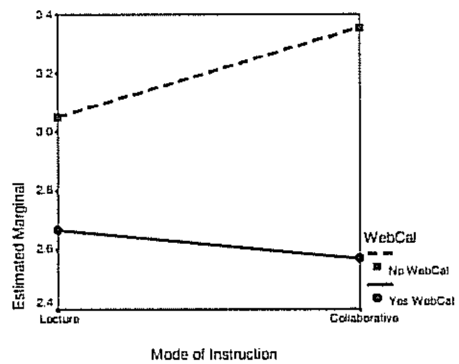
The graph above indicates that WebCal students studied more hours than non-WebCal students.

Table 15. The Workload

Mode of Instruction		Descriptive		
		Mean	St. Dev.	N
Lecture	non-WebCal	3.05	0.718	57
	WebCal	2.67	0.803	63
	Total	2.85	0.785	120
Collaborative	non-WebCal	3.35	0.915	31
	WebCal	2.57	0.714	70
	Total	2.81	0.857	101
Total	non-WebCal	3.16	0.801	88
	WebCal	2.62	0.756	133
	Total	2.83	0.817	221

Tests between subjects effects		
Source	F	Sig.
Mode of Instruction	0.901	0.344
WebCal Usage	28.749	0.000
Mode of Instruction*WebCal Usage	3.321	0.070

a R Squared = 0.121 (Adjusted R Squared = 0.109)



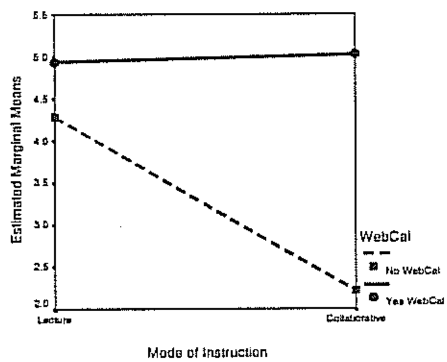
Consequently, WebCal students perceived the workload in Calculus to be significantly heavier (in comparison to other science courses) than their peers in non-WebCal classes who report that the workload in Calculus was lighter than the workload in other science classes ($p < 0.001$).

In addition, students' report on feelings of confusion they experienced in different instructional settings.

Table 16. Class Feeling

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-WebCal	2.33	1.139	57	Mode of Instruction	0.717	0.398
	WebCal	2.49	1.045	63	WebCal Usage	2.553	0.112
	Total	2.42	1.089	120	Mode of Instruction*WebCal Usage	0.336	0.563
Collaborative	non-WebCal	2.37	1.238	32			
	WebCal	2.71	1.079	70			
	Total	2.61	1.136	102			
Total	non-WebCal	2.35	1.169	89			
	WebCal	2.61	1.065	133			
	Total	2.50	1.112	222			

a R Squared = 0.019 (Adjusted R Squared = 0.006)



Although there are no significant differences between groups, the WebCal students appear to be more confused than the non-WebCal students.

The effectiveness of Maple usage.

To examine the effectiveness of Maple usage, we compared Maple Lecture and Collaborative sections with non-Maple Lecture and Collaborative sections, using a 2 x 2 factorial design. (Note that the term non-Maple sections does **not** include sections that used WebCal.) Given that three different instructors taught these sections, we tested for the impact of differences between instructors. One-way ANOVA was conducted, where the independent variable was the instructor and the dependent variable was evaluation of instruction. Recall that this latter variable is based on students' class attendance and their global assessment of instruction. We found significant differences amongst the instructors ($F = 25.837$, $p < 0.001$).

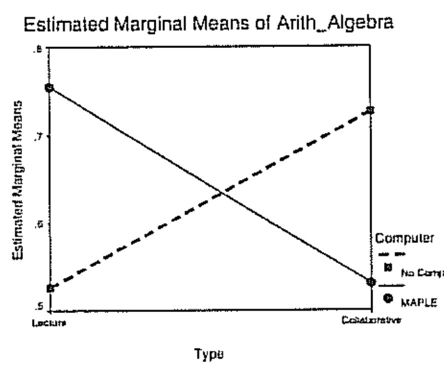
Given the above result, we needed to use a statistical method to correct for differences between instructors. Consequently, we used two-way ANCOVA to study whether there are significant differences between the means of outcome variables in various settings. The factors were: Mode of Instruction, with two levels (Lecture and Collaborative) and Maple Usage, with two levels (non-Maple and Maple), with the evaluation of instruction as the covariate. The results of the analysis presented below consist only of those that were significant. First, we focus on the impact on our measures of achievement, and then we show how Maple Usage and Mode of Instruction impact on student motivation to study mathematics.

Table 17 below provides descriptive statistics, mean, standard deviation (St. Dev.) and sample size in each cell, as well as totals for each (row, column) pair of the 2 x 2 factorial design, for the outcome variable arithmetic and algebraic skills. The last three columns of the table show the two main effects and any interaction between Mode of Instruction and Maple Usage. (The tables below have a similar structure.) As one can see in Table 17, there was one significant main effect: the students' performance on arithmetic and algebraic skills in Lecture and Collaborative sections was significantly different ($p < 0.001$). There was also a significant interaction between Mode of Instruction and Maple Usage ($p = 0.038$).

Table 17. Arithmetic and Algebraic Skills

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	0.53	0.347	25	Mode of Instruction	20.237	0.000
	Maple	0.76	0.551	8	Maple Usage	0.053	0.818
	Total	0.59	0.407	33	Mode of Instruction*Maple Usage	4.473	0.038
Collaborative	non-Maple	0.73	0.405	26			
	Maple	0.52	0.336	19			
	Total	0.64	0.388	45			
Total	non-Maple	0.63	0.387	51			
	Maple	0.59	0.415	27			
	Total	0.62	0.395	78			

[a]R Squared = 0.733 (Adjusted R Squared = 0.714)



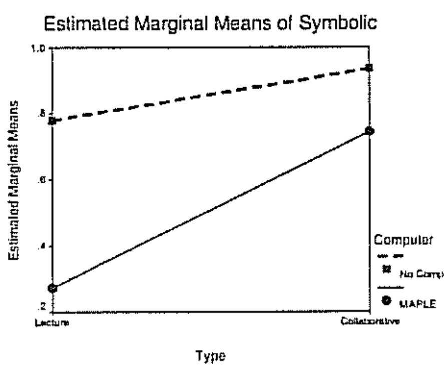
The graph above shows that while non-Maple students outperformed Maple student in Lecture sections, Maple students outperformed non-Maple students in Collaborative sections.

Table 18 below provides results for the outcome variable, use of symbolic language. The last three columns show that there were two significant main effects: students use of symbolic language in Lecture and Collaborative sections was significantly different ($p < 0.001$); and, non-Maple students use of symbolic language was significantly different ($p < 0.052$) from that of Maple students.

Table 18. Use of Symbolic Language

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	0.75	0.414	25	Mode of Instruction	14.886	0.000
	Maple	0.26	0.231	5	Maple Usage	3.919	0.052
	Total	0.67	0.428	30	Mode of Instruction*Maple Usage	1.140	0.289
Collaborative	non-Maple	0.93	0.677	27			
	Maple	0.80	0.336	19			
	Total	0.88	0.560	46			
Total	non-Maple	0.84	0.568	52			
	Maple	0.69	0.384	24			
	Total	0.79	0.519	76			

[a]R Squared = 0.733 (Adjusted R Squared = 0.714)



The graph above shows that Maple students outperform non-Maple students in the use of symbolic language. In addition, students in Collaborative sections outperform students in Lecture sections on this same measure of achievement.

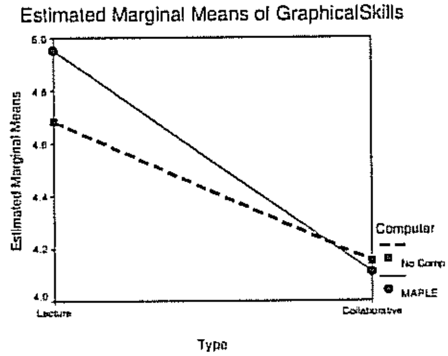
Table 19 below provides results for the outcome variable, understanding of graphs. From the table we see that there was only one significant main effect: students' performance on understanding of graphs in Lecture and in Collaborative sections was significantly different ($p < 0.001$).

Table 19. Understanding of Graphs

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	4.48	1.036	25	Mode of Instruction	67.616	0.000
	Maple	5.00	0.000	7	Maple Usage	0.041	0.840
	Total	4.59	0.937	32	Mode of Instruction*Maple Usage	0.208	0.650
Collaborative	non-Maple	4.08	1.640	24			
	Maple	4.45	1.301	19			
	Total	4.24	1.494	43			
Total	non-Maple	4.29	1.365	49			
	Maple	4.60	1.132	26			
	Total	4.39	1.290	75			

a R Squared = 0.928 (Adjusted R Squared = 0.923)

Estimated Marginal Means of GraphicalSkills



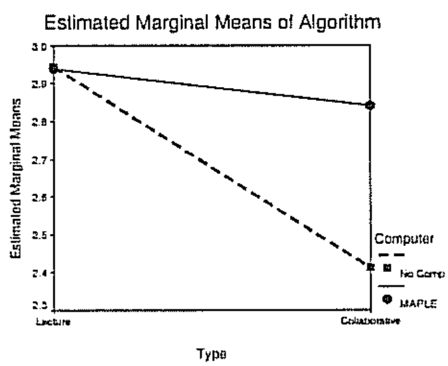
The graph above shows that students in Collaborative sections outperformed students in Lecture sections.

Table 20 below provides results for the outcome variable, use of algorithms. From the table we see that there was only one significant main effect: students' performance in the usage of algorithms was significantly different ($p < 0.001$) in Lecture and Collaborative sections.

Table 20. Use of Algorithm

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	2.83	0.698	25	Mode of Instruction	74.278	0.000
	Maple	2.96	0.901	10	Maple Usage	1.383	0.243
	Total	2.87	0.750	35	Mode of Instruction*Maple Usage	1.446	0.233
Collaborative	non-Maple	2.38	0.829	27			
	Maple	3.03	0.625	19			
	Total	2.65	0.811	46			
Total	non-Maple	2.59	0.794	52			
	Maple	3.01	0.716	29			
	Total	2.74	0.788	81			

[a]R Squared = 0.936 (Adjusted R Squared = 0.932)



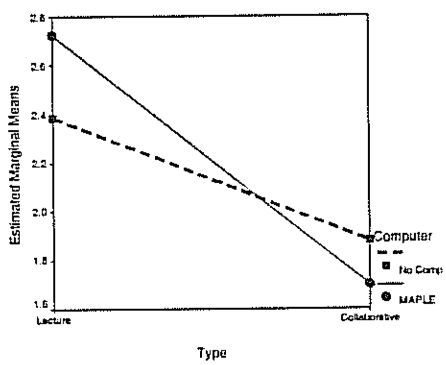
The graph above indicates that non-Maple students in Collaborative sections outperformed other students in correctly using algorithms. However, we note that this effect was not significant.

There were neither main effects nor an interaction effect in students' understanding of concepts. However, we determined that students' ability to obtain a correct answer was different across instructional settings. Table 21 below provides results for that outcome variable. From the table we see that there was only one significant main effect: students' ability to obtain a correct answer was significantly different ($p < 0.001$) in Lecture and Collaborative sections.

Table 21. Correct Answer

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	2.21	0.696	25	Mode of Instruction	50.330	0.000
	Maple	2.76	0.880	10	Maple Usage	0.050	0.833
	Total	2.37	0.781	35	Mode of Instruction*Maple Usage	2.130	0.149
Collaborative	non-Maple	1.83	0.807	27			
	Maple	1.99	0.714	19			
	Total	1.90	0.766	46			
Total	non-Maple	2.01	0.773	52			
	Maple	2.25	0.846	29			
	Total	2.10	0.803	81			

[a]R Squared = 0.900 (Adjusted R Squared = 0.893)



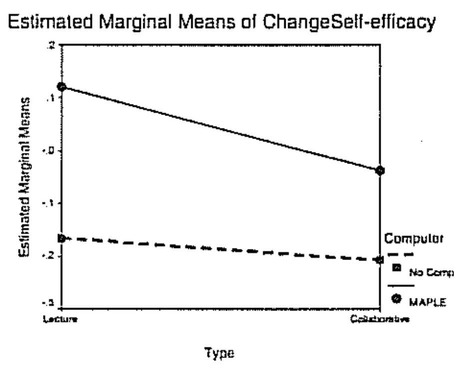
The graph above indicates that students in Collaborative sections outperformed students in Lecture sections in their ability to solve problems without making errors.

Next we examined the differential effects of instructional settings on students' motivation to study mathematics. There was a significant main effect ($p = 0.044$) on the self-efficacy scale. Table 22 below shows this result.

Table 22. The Change in Self-efficacy

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	-0.11	0.455	25	Mode of Instruction	0.894	0.347
	Maple	0.11	0.393	11	Maple Usage	4.195	0.044
	Total	-0.04	0.443	36	Mode of Instruction*Maple Usage	0.347	0.558
Collaborative	non-Maple	-0.19	0.403	26			
	Maple	-0.13	0.383	19			
	Total	-0.17	0.392	45			
Total	non-Maple	-0.15	0.427	51			
	Maple	-0.04	0.398	30			
	Total	-0.11	0.418	81			

[a]R Squared = 0.083 (Adjusted R Squared = 0.035)



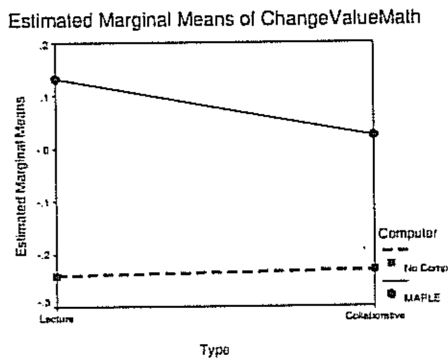
The graph above indicates that students' self-efficacy did not change much, but it also indicates that while non-Maple students became slightly more confident in themselves, Maple students' perception of self-efficacy decreased over the course of the semester.

We did not find significant differences across instructional setting in attitudes towards mathematics. However, we found a significant main effect on students' value of knowledge of mathematics ($p = 0.009$). Table 23 below gives this result.

Table 23. The change in students' value of knowledge of mathematics

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	-0.15	0.403	25	Mode of Instruction	0.206	0.651
	Maple	0.12	0.508	11	Maple Usage	7.293	0.009
	Total	-0.07	0.448	36	Mode of Instruction*Maple Usage	0.345	0.559
Collaborative	non-Maple	-0.20	0.480	27			
	Maple	-0.13	0.384	19			
	Total	-0.17	0.439	46			
Total	non-Maple	-0.18	0.441	52			
	Maple	-0.04	0.441	30			
	Total	-0.13	0.443	82			

[a]R Squared = 0.120 (Adjusted R Squared = 0.0740)



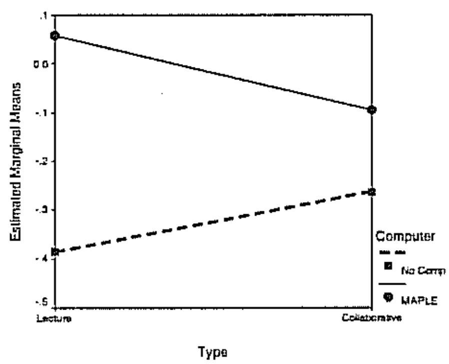
The graph above indicates that while non-Maple students valued knowledge of mathematics more than they did at the beginning of the semester, Maple students perception of the value of knowledge of mathematics decreased over the course of the semester.

We also determined that instructional settings had a significant impact on students' goal orientation. Table 24 below shows the result of this analysis.

Table 24. The Change in Students' Goal Orientation

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	-0.31	0.393	25	Mode of Instruction	0.032	0.858
	Maple	0.05	0.335	11	Maple Usage	10.715	0.002
	Total	-0.20	0.408	36	Mode of Instruction*Maple Usage	2.832	0.097
Collaborative	non-Maple	-0.25	0.326	26			
	Maple	-0.21	0.348	19			
	Total	-0.23	0.332	45			
Total	non-Maple	-0.29	0.358	51			
	Maple	-0.12	0.361	30			
	Total	-0.22	0.365	81			

[a]R Squared = 0.157 (Adjusted R Squared = 0.112)



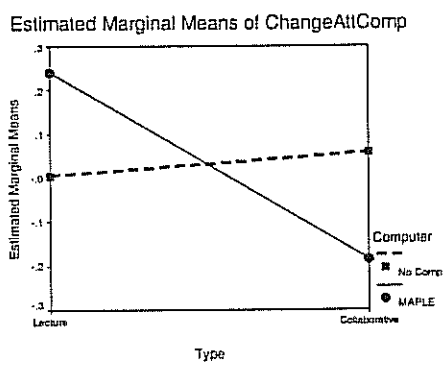
The graph above indicates that non-Maple students became more mastery oriented while Maple students became more performance oriented over the course of the term. This main effect is significant ($p = 0.009$).

Although the following results are unrelated to our hypotheses, we think that they are interesting to note. There was a significant interaction effect on changes in students' attitudes towards computers as a learning tool ($p = 0.037$). Table 25 below shows the result of this analysis.

Table 25. The Changes in Students' attitudes towards Computers

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	0.03	0.519	25	Mode of Instruction	1.368	0.261
	Maple	0.24	0.405	11	Maple Usage	0.044	0.835
	Total	0.09	0.492	36	Mode of Instruction*Maple Usage	4.499	0.037
Collaborative	non-Maple	0.06	0.445	26			
	Maple	-0.22	0.478	19			
	Total	-0.06	0.475	45			
Total	non-Maple	0.04	0.478	51			
	Maple	-0.05	0.498	30			
	Total	0.01	0.485	81			

[a]R Squared = 0.090 (Adjusted R Squared = 0.030)



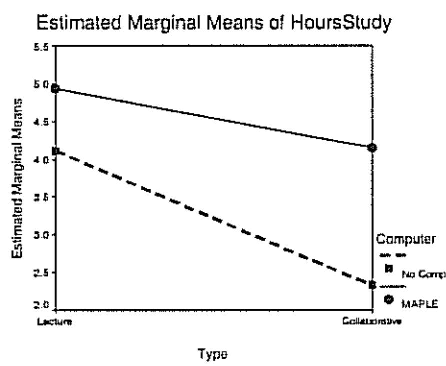
The graph above indicates that Maple students became more negative about using computers in Lecture sections and more positive about it in Collaborative sections. Non-Maple students attitude was the same in both Lecture and Collaborative sections.

The post-questionnaire asked students the number of hours they studied mathematics (Study Hours), how they perceived the course workload (Workload) and whether they were confused when leaving a class room (Class Feeling). Although these results were not significant, we include them in our discussion because they may be related to changes in students' attitudes towards using computers as learning tools.

Table 26. Study Hours

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	4.32	5.800	25	Mode of Instruction	1.641	0.204
	Maple	4.95	3.253	10	Maple Usage	1.684	0.199
	Total	4.50	5.161	35	Mode of Instruction*Maple Usage	0.296	0.588
Collaborative	non-Maple	2.36	1.531	25			
	Maple	3.81	1.816	18			
	Total	2.97	1.788	43			
Total	non-Maple	3.34	4.314	50			
	Maple	4.21	2.432	28			
	Total	3.65	3.754	78			

[a]R Squared = 0.069 (Adjusted R Squared = 0.0180)

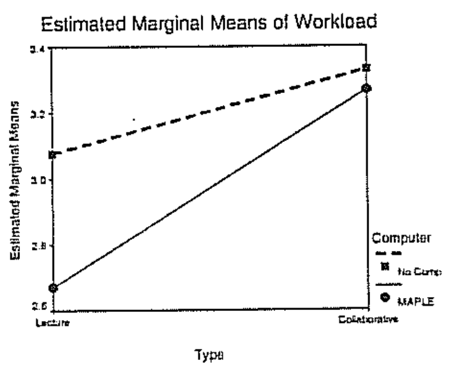


It should come as no surprise that there was a significant main effect of instructional setting on how students compare their workload in Calculus to that of other science courses. There was a significant difference between students in Lecture sections and students in Collaborative sections ($p = 0.042$). Table 27 below shows the result of this analysis.

Table 27. The Workload

Mode of Instruction		Descriptive			Tests between subjects effects		
		Mean	St. Dev.	N	Source	F	Sig.
Lecture	non-Maple	3.24	0.779	25	Mode of Instruction	4.261	0.042
	Maple	2.64	0.674	11	Maple Usage	1.133	0.291
	Total	3.06	0.791	36	Mode of Instruction*Maple Usage	0.806	0.372
Collaborative	non-Maple	3.37	0.926	27			
	Maple	3.00	0.840	18			
	Total	3.22	0.902	45			
Total	non-Maple	3.31	0.853	52			
	Maple	2.86	0.789	29			
	Total	3.15	0.853	81			

[a]R Squared = 0.144 (Adjusted R Squared = 0.0990)



The graph above indicates that students in Lecture sections thought the course was heavier than students in Collaborative sections.

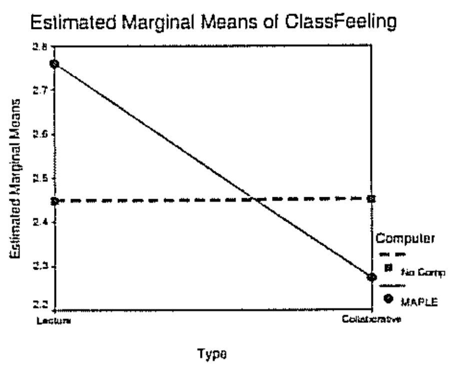
There were no significant differences amongst the different instructional settings in terms of students' feelings of confusion upon exiting class. Table 28 below shows the results of this analysis.

Table 28. Class Feeling

Mode of Instruction		Descriptive		
		Mean	St. Dev.	N
Lecture	non-Maple	2.04	1.020	25
	Maple	2.82	1.401	11
	Total	2.28	1.186	36
Collaborative	non-Maple	2.33	1.209	27
	Maple	2.95	0.911	19
	Total	2.59	1.127	46
Total	non-Maple	2.19	1.121	52
	Maple	2.90	1.094	30
	Total	2.45	1.156	82

Tests between subjects effects		
Source	F	Sig.
Mode of Instruction	0.914	0.342
Maple Usage	0.058	0.810
Mode of Instruction*Maple Usage	1.074	0.303

[a]R Squared = 0.296 (Adjusted R Squared = 0.260)



Discussion

The first hypothesis in this study concerns students' performance and changes in motivation to study mathematics. In terms of performance we hypothesized that students using WebCal students would outperform non-WebCal students. From amongst all six measures of achievement we have a significant difference between results in WebCal and non-WebCal sections in only two measures, Understanding of Algorithms and Understanding of Graphs. In the first of these two measures, Understanding of Algorithms, the WebCal students out performed their counterparts ($p = 0.006$). Knowledge of algorithms is both an essential component of mathematical knowledge and a higher level cognitive skill. It involves using the conditional knowledge of when/when not to use a given algorithm to solve a problem. Further, students must possess the ability to adapt such algorithms to suit the demands of particular problems. Thus, we can say that there is a very high probability that WebCal students' mathematical knowledge is higher than that of non-WebCal students. We attribute this highly significant effect to the impact of student tasks in WebCal sections (see Appendix 6). Students were asked to explain and justify steps they used in problem solving. In both groups, WebCal and non-WebCal, the scores of students in Collaborative sections tended to be better than of those in Lecture sections ($p = 0.064$). We understand this to mean that discussing with peers probably helps students to understand algorithms, but using WebCal (versus not using WebCal) helps even more.

In the second significant measure, Understanding of Graphs, the WebCal students again out performed their counterparts ($p < 0.001$). Note that the coded student solutions for problems P13 and P14 (see Appendix 3) are two major components of this measure. Most Calculus instructors would agree that these two problems represent a synthesis of understanding of the concepts of limits and derivatives. In addition, a student who successfully solves these two problems has to be able to transfer the data that were gathered in a symbolic language, using an particular set of algorithms, into a graphical perspective. Thus, it is reasonable to say that Understanding of Graphs represent a high level of skill (algorithmic, synthesis of concepts, and transfer of perspectives). Students who score high on this variable have demonstrated a good conceptual understanding of the central concepts of Calculus I. Thus, highly significant results on this variable indicate that use of WebCal had a very positive impact on students performance, as predicted in the hypothesis. In both groups, WebCal and non-WebCal, the scores of Collaborative section students were also significantly better ($p = 0.034$) than of those in Lecture sections. We understand this to mean that both discussing with peers and using WebCal both help students to understand graphs better.

From the results section above, we note that there was one further significant main effect, namely the effect of Mode of Instruction on obtaining correct answers. Here the Collaborative sections performed significantly better ($p = 0.009$) than the Lecture sections. There was no significant interaction between Mode of Instruction and WebCal Usage. Thus, in terms of obtaining correct answers, the collaborative learning aspect of an instructional setting seems to be more important to student success than WebCal Usage. Probably it is classroom discussion with peers that helps students to learn to work through problems to the very end without errors. It is also conceivable that having collaborated in the classroom, students become more likely to collaborate outside of the classroom, and thus are learning from their peers in an ongoing manner.

In addition to the above main effects, we note that in a Lecture setting students in WebCal sections outperform, both on arithmetic/algebraic skills as well as in use of symbolic notation, the comparable students in non-WebCal sections. This trend reverses significantly ($p = 0.009$) for arithmetic/algebraic skill when we examine Collaborative settings. Similar trends are observed in the case of use of symbolic notation, although these trends are less significant ($p = 0.080$). Here it seems that interaction between WebCal usage and Lecture mode of instruction have a positive impact on students' performance in both arithmetic/algebraic skills and use of symbolic notation. This result contradicts our hypothesis that there would be a positive interaction between using WebCal and being in a Collaborative sections that would increase students' performance on these two variables.

It is important to say that although both arithmetic/algebraic skills and use of symbolic language are necessary conditions for successful mathematical problem solving, they represent a low level of skills. To use an analogy, these skills are to being a mathematician as spelling is to being a writer. Both of these skills are largely gained by individuals through repetition. In the WebCal sections collaborative tasks tended to focus on reasoning, and practice shifting between perspectives (see Appendix 6), whereas in the non-WebCal sections collaborative tasks focussed more on traditional problems that heavily involve arithmetic/algebra and use of symbolic notation. Thus, in the non-WebCal sections, students in Collaborative settings were provided with supervised practice in these skills, hence their performance was better than the non-WebCal section students in Lecture settings. However, in the WebCal sections, students in Collaborative settings were not provided with much practice at these skills, and, understanding that the focus was on reasoning and shifting between perspectives, they probably relied on the most skilled group member to carry out the simpler tasks, leaving themselves under practised. We note that despite this result, overall there was no significant difference between WebCal and non-Webcal sections on these two measures of achievement so WebCal students performance in these areas did not decline.

We anticipated that students' motivation to study mathematics would improve as result of using WebCal. We studied changes in student motivation (self-efficacy, attitude towards mathematics and value of knowledge of mathematics) and found that there were no significant differences between the changes of motivation amongst

WebCal and non-WebCal students, nor were they any significant differences between students in Lecture sections and in Collaborative sections, nor did we observe any significant interaction effects. In some sense this is an unexpected result. Many studies on collaborative learning have shown a positive impact of this instructional design on student motivation (*e.g.,* Lou *et al.,* 2002).

Although we did not formulate any hypothesis concerning students' preference for a structured learning environment we tested whether there was a significant difference between WebCal and non-WebCal students concerning change in this variable. We found that WebCal students became more independent learners over the course of the term, while non-WebCal students became more dependent learners. The difference between these two results is highly significant, both statistically ($p = 0.002$) and in its implications for the likelihood of such students becoming life-long learners. Since creating life-long learners is an important goal in science education, or for that matter in all educational fields, this is very good news.

Although we did not anticipate that WebCal usage would have any impact on how hard students study, we noted when looking at the variable WebCal Usage that students reported significantly more ($p = 0.003$) hours of study in sections that used WebCal versus sections that didn't. We further noted that there was a significant interaction ($p = 0.062$) between the variable WebCal Usage and the variable Mode of Instruction. That is, students in sections using WebCal reported similar numbers for Hours of Study in Lecture sections versus Collaborative sections. However, students in sections not using WebCal reported a drop in the Hours of Study from a Lecture section to a Collaborative section.

When we looked at student perception of Workload (in comparison to other science courses), our results were similar. That is, students reported a significantly higher ($p < 0.001$) Workload for sections using Webcal versus sections that don't. In addition, there was once again an interaction ($p = 0.070$) between the variable WebCal Usage and the variable Mode of Instruction, but it seems reversed. That is, students in sections using WebCal reported a similar Workload in Lecture sections versus Collaborative sections. However, students in sections not using WebCal reported a larger workload in Collaborative sections over Lecture sections. To put this briefly, students in sections using WebCal reported working longer and perceived that they are working harder than their non-WebCal counterparts, but WebCal students did not find that working collaboratively changed their workload, while non-WebCal students did.

Note that none of the results that are discussed below were significant, however they are sufficiently interesting to warrant discussion. Students generally tended to be less confused when leaving class in Lecture sections versus Collaborative sections. This makes sense in that a lecture consists of a teacher, having organized her thoughts in a particular area, presenting those organized thoughts. However, collaborative group work involves students discussing ideas with other students and/or with teachers, and so students have had to confront and often not fully resolve many questions. We also noted that WebCal sections report more confusion than non-WebCal sections, and again this would seem to be a natural result of students frequently doing in-class explorations and confronting but not fully resolving many questions. Conceptual change theory (Posner et al, 1982) posits that cognitive dissonance is one of the cognitive conditions required for achieving conceptual change. However, Redish (Redish, 2003) points out that for some students such dissonance may make them feel incompetent, thus decreasing their motivation, which in turn may lead to lower performance. We can relate this finding to changes in students' attitudes towards computers. Students in Lecture sections generally improved their attitude towards using computers, while students' positive attitude towards computers declined in Collaborative sections ($p = 0.047$). It is possible that this decline is related to frustrations WebCal students may have experienced.

The third hypothesis in this study proposes that students using Maple in Collaborative Mode of Instruction sections will outperform students using Maple in Lecture Mode of Instruction sections as well as students who did not use Maple materials in either Lecture Mode of Instruction or Collaborative Mode of Instruction sections. There are no significant differences between the results in Maple sections versus the results in non-Maple sections in all but one measure of performance. Maple students significantly outperformed non-Maple students in their use of symbolic notation ($p = 0.052$). This result is most likely an indication that working with Maple focuses students attention on the syntax of Maple's mathematical/computer language and this in turn transfers to significantly better use of mathematical symbolic language. Appropriate use of symbolic language

In addition, there is a significant interaction between Maple and Mode of Instruction ($p = 0.038$) on the arithmetic/algebraic skill subscale. While students in non-Maple Lecture sections outperform students in Maple lecture sections, this trend reverses for students in Collaborative sections. Perhaps this is because in the non-Maple Lecture sections, the students watch the teacher perform arithmetic/algebra on the board more frequently than in Maple Lecture sections, where the teacher uses Maple to avoid some of this work.

The impact of Maple usage on student motivation merits further investigation. The results indicate that both students' self-efficacy in mathematics and their perception of the value of mathematics in their future decreased over the course of the semester. These results may be related to the fact that Maple V version 5.1 was used in the course. This version of Maple is rather difficult to master and hence it is likely that many students had difficulties with it. That is, the awkwardness of Maple's interface probably caused such students to feel less competent and confident in their studies of Calculus. We note that Maple students became more performance oriented learners over the course of the term. This finding supports our conjecture. Students who were struggling with the interface may have abandoned an effort to master Calculus. At the end all they cared about were their grades. It is important to collect more data to determine both why students felt that way and how the instruction should be modified to help students. Student motivation to study mathematics impacts on their future success. It may also have a serious impact on students' choice of a future career because the likelihood of them continuing to take courses in mathematics has decreased.

We note that when looking at the variable Maple Usage, there is no significant difference for Hours of Study or Workload. Further, when looking at the variable Mode of Instruction, there is no significant difference for Hours of Study. However, the students in Collaborative sections perceive that they have a significantly lighter ($p = 0.042$) workload than their counterparts in Lecture sections. We note that there is a similar trend in the Hours of Study, but it is not significant. Also, although it is not significant for either of the dependent variables, we note that students in Maple sections report studying longer and perceive that they have a heavier workload. This trend is similar to one noted above for students in WebCal sections.

There are no significant results involving the variable feelings of confusion but we observe an interesting trend. Students in Maple sections generally tended to be more confused when leaving classes in Lecture sections than in Collaborative sections, while students in non-Maple sections appear to have equivalent feelings of confusion whether in Lecture or Collaborative sections. Given that group activities were limited to eight classes per semester, Maple and non-Maple sections, this latter result makes more sense. That is, whether a section was Lecture or Collaborative, most of the time teachers were presenting thoughts that they had previously organized, and so the feelings of confusion should be the same. However, in Maple sections the teacher used Maple about 15% of the time to explain or demonstrate concepts, with this in addition to the eight student Maple tasks, done either individually or collaboratively. We note that students' attitude towards using computers as a learning tool became more positive in Maple Collaborative sections, while students' attitude towards using computers as a learning tool became more negative over the course of the semester in Maple Lecture sections. This result is statistically significant ($p = 0.037$). We speculate that collaborative group work with Maple increased students' familiarity, hence comfort level, with Maple, and so these students found classroom explanations involving Maple helpful in increasing their understanding. However, since the version of Maple used, Maple V version 5.1, is not particularly user friendly, students must truly master a complex syntax if they are to avoid frustrating errors. Thus, students working individually on the eight Maple labs probably never reached the same comfort level with the software as students working in groups. Consequently, they would have felt that using Maple to explain concepts was confusing, not helpful.

We note that we did not have any significant differences on the variable Understanding of Concepts. This is largely due to insufficient data. As we stated earlier we had to exclude a large number of problems from coding because instructors did not follow the research protocol when including these problems in examinations. In addition, as we also stated earlier, the initial set of twenty one problems was negotiated as a set of problems that all instructors were willing to use in their examinations. Researchers attempted to include in the set a number of problems that might have probed students' conceptual understanding at greater depth. Many of these suggested problem included requiring students' to write explanations. Unfortunately, some of the instructors who participated in the project rejected such problems because they felt that their students would not be adequately prepared to answer them. This is an issue that many classroom studies have to deal with. Usually the focus of an educational innovation is to change student learning, either by improving performance in what is currently learned or changing the depth or very nature of what is learned. In the latter case, the outcome of such change cannot be tested using an experimental/control design of research because teachers of control sections will not test students in areas that they are not attempting to teach and do not believe students capable of mastering.

Conclusion

In conclusion, this research has demonstrated that using WebCal has an overall positive impact on student learning of Calculus. In some areas this impact is clearly positive, while in other areas there is no significant impact. A commonly voiced concern among mathematics instructors is that integrating the use of computers into mathematics education will have a negative impact on students' acquisition of basic skills. On the other hand, proponents of the use of computers often focus on the benefits of such use on student learning. This research project contributed to this ongoing debate.

The fact that WebCal students did just as well as non-WebCal students on skill tasks (arithmetic/algebraic and use of symbolic language) should satisfy those people concerned with the potential "loss of skills". At the same time, the fact that WebCal students outperformed their counterparts on some higher level skills should convince some instructors to incorporate some or all of the instructional design characteristics of our WebCal implementation. These results are somewhat less convincing in terms of Maple usage. This study also demonstrated the importance of including collaborative learning strategies alongside the incorporation of the use of technology in instructional designs. While using WebCal improved student understanding of graphs (a higher level skill), the students in collaborative sections were more capable of solving problems without making errors. Finally, this study also demonstrated that students need, and then benefit, when frequent elaborative feedback is provided during the learning process. We have also shown that the provision of individual elaborative feedback while students are learning is feasible in a computer laboratory while it is practically impossible in traditional classrooms.

Numerous research issues impacted on this quasi-experimental study. Not only is it impossible to randomly assign participants into treatments, but it is also extremely difficult to control the number of participants in each treatment. We began with the assumption that Calculus I classes have an average of forty students. We gathered data from both Fall 2001 and Fall 2002 to increase the number of students in each treatment. However, student enrollment in one of our treatments was low in both semesters, and therefore the number of participants per cell was not the same across the treatments, leading to a large decrease in statistical power. Because of this problem we could not investigate some of the questions that we had originally planned when designing this experiment.

We have found that students reported working on average 20% more study hours in WebCal sections than in non-WebCal sections. When considering the time spent by the instructors preparing to teach Calculus using WebCal we suspect that they also felt that they worked much harder. We did not collect any data on these issues but it seems worth further investigation. While it is well known that "time on task" is the most important factor in learning, a too heavy workload may also discourage students from taking courses that integrate technology. We anticipate that this issue of both teacher and student workload is apt to influence how and whether technology will be used in CÉGEP mathematics classrooms in the near future. To investigate this issue one needs to focus on two aspects: transparency and ease of use of technology; and, the need to develop student tasks that are specific to technology based settings and different from those normally used by mathematics instructors. Both of these aspects appear to have played a role in the Maple implementation. We also have anecdotal evidence that some students were frustrated with LiveMath. (During the period of the research project the ownership of this software changed hands and Microsoft changed the plugin architecture of its browser, Internet Explorer. As a result, maintenance of the free LiveMath plugin deteriorated to the point that some students never managed to be able to install it at home, a clear discouragement to them learning through experimentation). Since site licensing of the current version of Maple, Maple 9.xx, has now been offered to CÉGEPs, and this version allows instructors to create self-standing Java applets with custom interfaces, perhaps student difficulties can be addressed in the near future. We certainly plan to investigate these issues in future research and development projects.

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Discovery Learning Using Computer Simulations: Does Feedback Increase Student Performance [1]

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Abstract

In this research we have studied feedback and how it affects student performance in discovery learning instructional settings enhanced by computer simulations. We examined two aspects of feedback: timing - whether feedback is provided during or only after learning activities; and type, specifically "single-try" elaborative feedback versus "multiple-try" elaborative feedback. We have found that: average students improve their verbal reasoning and graphical skills when exposed to "single-try" elaborative feedback that is provided during experimentation with simulations; failing students make gains only in their verbal reasoning; and, feedback has no effect on the performance of high achievers.

Keywords: mathematics education, post-secondary education; feedback; computer simulations

Introduction

A report by Hodgson [1] sounded an alarm among mathematics instructors. The findings indicated that students in traditional lecture-based courses develop a narrow algebraic perspective of concepts while it is known that experts move fluidly between the verbal, graphical, numerical and algebraic perspectives in solving problems. In lecture-based courses teachers indeed do model verbal reasoning for the benefit of students, but since the students are judged predominantly on the basis of symbolic work, students do not adequately learn to reason in words. Graphical and numerical activities are often neglected in lectures because they are tiresome and time consuming, so students are often unable to reason in those terms as well.

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[1] This is an intact version of a research paper that has been already published (Dedic & Rosenfield, 2003).

Many attempts to improve students' conceptual understanding of mathematics involve instructional designs that use discovery learning with computer simulations, often in combination with cooperative learning. These designs are based on a constructivist perspective of learning through active exploration of concepts. Working with peers while experimenting with simulations can provide an environment in which all four perspectives may be exercised. Computer simulations allow learners to generate a large number of "instances" of a phenomenon, from both a graphical and numerical perspective, in a short period of time. Papert [2] proposed that when students experience a phenomenon sufficiently often they begin to see a pattern. This allows them to internally generate a rule explaining the phenomenon (hypothesize). Once a hypothesis has been formed, further use of simulations facilitate its testing. The problem is that students often do not know how to experiment [3]. This may explain why research to date has not been able to convincingly demonstrate that student achievement in courses that use these instructional settings is significantly better than student achievement in traditional courses.

In their review of effectiveness of discovery learning with simulations, de Jong and van Joolingen [4] describe four categories of skills that learners need to possess in order to benefit from discovery learning. Students need to be able to: generate a hypothesis; design experiments to test it; interpret experimental data; and regulate the discovery process. In a qualitative study of student perceptions of their learning through discovery with computer simulations, we interviewed nine students over one semester in a course on Differential Calculus [5]. We found among other things that students had difficulty remembering and interpreting graphical data generated via simulation. Thus, their ability to formulate and test hypotheses was compromised [4].

There may be other reasons why learning with computer simulations may not improve learning outcomes *e.g.,* their motivation may be affected because they may miss the feedback provided by teachers. Learners may not benefit from work with simulations because they concentrate on the completion of an activity ("product orientation"), as opposed to focussing and reflecting on the underlying principles ("process orientation") [6].

A number of recent studies have shown that when students are helped in developing these inquiry skills, then discovery learning with computer simulations is successful (e.g., [7], [8]). In their study in physics education, Rieber et al. [6] found that providing elaborative feedback with simulations is more effective than experimentation with simulations alone. They speculated that feedback helps students to focus their attention away from the seductive details of routine computation towards more difficult cognitive activities such as hypothesis formulation and testing. Alternatively, in this study feedback provided during student experimentation with simulations may have operated just as the learning theories predict. Reviews of studies have demonstrated that immediate and frequent feedback improves learning (e.g., [9], [10]).

The literature on distance learning environments shows that feelings of isolation and the absence of the kind of feedback often obtained through a smile or a nod in traditional classrooms leads to failure. Despite feedback in the form of grades, students feel isolated and uncertain about their learning when working with computer simulation in Calculus[5]. Recent improvements in technology make it possible to provide feedback on students' screens while they are experimenting with simulations. This may lessen the need for teacher generated feedback. It may also help self-regulated learners to monitor their progress [11], and it may also reassure uncertain students that they are learning while they experiment. Incorporating such formative feedback into computer simulations may be effective for a number of other reasons. First, it may guide cognitive activities during which knowledge is accreted, tuned, and restructured [12]. In lecture-based courses feedback is usually provided through summative evaluation after learning takes place. Since the focus of such feedback is on the product of learning it may fail to guide the process of learning. Secondly, it may benefit students by helping them to focus on hypothesis generation and testing, as well as on improving their skills in experimentation.

The objective of this study was to evaluate an instructional design that promotes the development of graphical and verbal perspectives on central concepts of Calculus. We hypothesized that students who receive elaborative feedback while they experiment with simulations will perform better on graphical and verbal tasks then those students who receive the elaborative feedback only after the learning activity. We also wanted to investigate which type of feedback is more efficient in promoting the development of verbal and graphical perspectives of mathematical concepts.

Methodology

Instructional Context:

The experiment was carried out in a college level course in Differential Calculus. On-line course materials, WebCal, which inculcate the "Rule of Four", namely that students need to develop an understanding and linkage between four perspectives on any mathematical concept (symbolic, numeric, graphical and verbal) were used in addition to a textbook. Classes were held in a computer laboratory with two students per computer. Each lesson consisted of a series short lecture presentations, which included computer simulations as demonstrations of concepts presented, interspersed with student driven experimentation via simulations. For example, in one of the simulations students are meant to observe a tangent line move from left to right along the graph of a function and are expected to make verbal predictions about the graph of the derivative of the function.

Intervention Structure:

For the purposes of this intervention two classes were each divided into two groups so that each student worked alone on their own computer. A 90-minute lesson on the relationship between $f(x)$, $f'(x)$ and $f''(x)$ began with a 20-minute instructor explanation, structured around an animated simulation. Students then worked individually for 25 minutes on each of two computer simulations. The first simulation focussed on how to analyse a given $f(x)$ graph so as to predict the shape of a graph of $f'(x)$, while the second one focussed on how to analyse a given $f'(x)$ graph so as to predict the shape of a graph of $f(x)$. During each simulation two formative pop-up quizzes would appear on the screen (except for control group students. The first one was timed to pop-up seven minutes after the individual student began to work, and the second one after fifteen minutes. The individual work session was followed by a short (15-minute) formative paper-based quiz designed to help students to self-assess their mastery of this topic. Students then had two days to complete an assignment on this topic, with the solution for this assignment posted on-line immediately after the submission time. One week after the lesson, as a post-test, a summative quiz was administered during regular class time.

Intervention Materials (embedded formative quizzes):

Feedback research shows that elaborate feedback is generally found to be more effective than other simpler forms of feedback such as "right" or "wrong" or "correct answer is ..." ([9], [13], [14]). Another characterization of feedback is based on the number of chances that students are given to respond correctly to a question. Reports on the effectiveness of "answer-until-correct", or its variant, "multiple-try feedback", are mixed [15]. However, Clariana's review [14] of the literature reveals a number of studies in which the effectiveness of "multiple-try-feedback" is found to be superior to simple feedback. Learner frustration with "multiple-try-feedback" is posited by Clariana [14] as an explanation for its ineffectiveness when used with low performing students, while high performing students have been shown to prefer it. But a recent study shows that "multiple-try-feedback", when combined with elaborative feedback, can be more effective than “single-try-feedback” [16].

Based on this body of knowledge the conditions regulating feedback were as follows: there were two settings for the timing of elaborative feedback in this study. All students were given feedback during the formative paper-based quiz at the end of the lesson (via an associated pop-up question); some students were also given feedback during the period of experimentation prior to the end of lesson quiz (via two pairs of pop-up quizzes). In addition, there were two types of feedback in this study. Some students were given a single opportunity (“single-try- feedback” condition) to answer correctly, with an incorrect answer eliciting feedback that explained the correct answer. Other students were given two opportunities (“multiple-try-feedback” condition) to arrive at the correct answer. In this case a first incorrect answer elicited feedback explaining presumed reasoning error(s) associated with their incorrect response, followed by an invitation to try again. If a second incorrect answer was given, then the correct answer, complete with explanation, was presented.

To illustrate the nature of the questions and the feedback provided, below there is one question from a pop-up quiz, along with the response given to the correct answer, and for one of the incorrect answers.

The four statements a ... d below describe relationships between the functions f and f' that we use when we sketch a graph of f', given one of f. One or more of these statements may be false. Read each statement carefully and decide whether or not it is true or false. Then click the button beside the answer that agrees with your evaluation of the truth/falsehood of all four statements.

a. Slopes of tangent lines to a graph of f are y-coordinates in a graph of f'.
b. If the slope of a tangent line to a graph of f is positive at $x = a$, then the slope of a tangent line to the graph of f' at the same x will be positive.
c. Slopes of tangent lines to a graph of f' are negative throughout any x - interval where a graph of f is concave down.
d: At a value of x where a graph of f changes concavity, the slope of a tangent line to a graph of f' will be zero.

1. all of the above are true
2. a, b and c are true, d is false
3. b, c and d are true, a is false
4. a and c are true, b and d are false
5. a, c and d are true, b is false

Answer for 5 (which is correct): The statements a, c and d are true. The statement b is the only false statement. The slope of the tangent line to a graph of f does not tell us anything about the slope of a tangent line to a graph of f', but as a correctly states, it tells us about a y-coordinate value of the graph of f'. The sign of the slope of a tangent line to a graph of f' graph is indicated by the concavity of the f graph. Please continue to work with the LiveMath insert to increase your understanding of the relationships between f, f' and f''.

Answer for 2 (which is not correct): Your answer is incorrect because statement d is true. On an interval where a graph of f is concave up, we notice that the slopes of tangent lines to that graph increase in value as we move across the graph, left to right. But slopes of tangent lines to a graph of f are actually y-coordinates of a graph of f', so on that same interval a graph of f' is increasing. Similarly, on an interval where a graph of f is concave down, we notice that the slopes of tangent lines to that graph decrease in value as we move across the graph, left to right, and so a graph of f' is decreasing. Thus, at a point where a graph of f changes concavity, the slopes of tangent lines to that graph change from increasing to decreasing (or from decreasing to increasing), which means that a graph of f' changes direction, so slopes of that graph of f' change sign. According to the Intermediate Value Theorem such changes in sign only occur at a zero of slope. Before indicating a different choice, please try to re-evaluate the truth/falsehood of each statement.

The two pop-up quizzes that were displayed on the screen while students experimented had different objectives. The first quiz focussed on the process of experimentation with simulation. It aimed to help students who did not have the appropriate inquiry skills or to reassure those who were proceeding appropriately but were uncertain. The second one focussed on the features of graphs that students must be able to recognize in order to accomplish the task. It aimed to help students who could not interpret graphical data collected in their experiments.

Participants:

One hundred and eight first year college students attending a junior college participated in this experiment. All students were registered in a Differential Calculus course that is compulsory in the pre-university Science Program.

Design:

Based on prior performance (as measured by performance on mid-term class tests) students were first grouped into three strata (n=36 in each strata): high achievers (with grades better than 74%; median = 81%); average students (with grades between 60% and 74%; median = 67%) and failing students (with grades less than 60%; median = 51%). Students in each of these three groups were then randomly assigned to one of the four conditions: single try feedback in the formative quiz at the end of the lesson; multiple try feedback in the formative quiz at the end of the lesson; single try feedback both during the experimentation with the two simulations and in the formative quiz at the end of the lesson; and, multiple try feedback both during the experimentation with the two simulations and in the formative quiz at the end of the lesson. Consequently, there were nine participants in each cell in this 3 x 2 x 2 design.

Student understanding of the relationship between $f(x)$, $f'(x)$ and $f''(x)$ was assessed on the basis of their performance on a post-test quiz administered one week after the lesson. This quiz consisted of a graph of a function $f'(x)$. From the features of the graph students were asked to sketch a graph of a function $f(x)$ and to explain verbally each step that they took in generating the graph of $f(x)$. The two dependent variables in this study are: GRAPH, measured by errors made in graphing; and, VERBAL REASONING, measured by errors made in explanations. Each of these two variables assesses student understanding of the relationship between $f(x)$, $f'(x)$ and $f''(x)$, but from two different perspectives. Two coders independently coded the appearance of graphs and verbal reasoning, and the agreement between their scores was over 90%. In all cases of disagreement it was shown that one of the coders had not used the coding schema appropriately.

Table 1

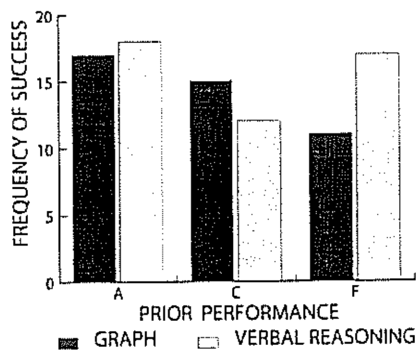
Ind. var.	Description
PRIOR PERFORMANCE	F (failing students), C (average students) and A (high achievers)
TIMING OF THE FEEDBACK	0 (feedback was provided only in the formative quiz) and 1 (feedback was provided both during the work with two simulations and in the formative quiz)
TYPE OF THE FEEDBACK	0 (multiple-try feedback) and 1 (single try feedback)
Dep. var.	**Description**
GRAPH	0 (a failure indicates one or more errors in the graph) and 1 (a success indicates a correct graph)
VERBAL REASONING	0 (a failure indicating that the explanations were either incomplete or incorrect or both) and 1 (a success indicating a correct explanation of all steps in solving the problem)

We used categorical variables and logistic regression to analyse the data. Table 1 shows the variables used in this experiment.

Results

The results show that 44 students (41% of the sample population) generated a flawless graph. An even higher proportion of students (47 or 44% of the sample population) produced a complete and correct verbal argument that guided their graphing of $f(x)$. A number of students reasoned correctly but failed to include every step. For example, of the thirty-five students who were coded as having made exactly one error, only nine actually made an error in their reasoning. Each of the remaining twenty-six students failed to mention one step.

The frequency of success in both GRAPH and VERBAL REASONING is shown on the chart below. It seems to suggest that student performance depends on PRIOR PERFORMANCE. The logistic regression shown below suggests otherwise.



The only significant result of the logistic regression between each of the independent variables and each of the dependent variables for the sample shows students' likelihood of success in VERBAL REASONING increases when the feedback is given during the experimentation as well as on the formative quiz (a positive coefficient of 0.752). The t-ratio is 1.904 and chi-square of 3.692 is significant at 0.055 level. No other model significantly improved this result. There is no effect of either TIMING OF FEEDBACK or TYPE OF FEEDBACK on likelihood of success in GRAPH.

We did not wish to pursue this approach much further, since it would force the introduction of more terms than we really needed. Since prior performance is introduced essentially as a control device (that is we are more interested in eliminating its effect than in actually studying it) we decided that a simpler and more efficient way of controlling for this variable is to study each performance group separately. That is we split the sample into three groups according to their prior performance and investigated the effects of TIMING OF FEEDBACK and the TYPE OF FEEDBACK for these groups separately. We found that neither the timing of the feedback nor the type of feedback increased the likelihood of success amongst the high achievers. The results of the logistic regression for average students is shown in Table 2 below.

The results from Model 1 indicate that the likelihood of correct verbal reasoning increases when feedback is given during the experimentation. The coefficient on the timing of feedback variable has a t-ratio equal to 1.619 which is significant at 0.10 level. The model correctly predicts 67% of responses. Model 2 includes an additional explanatory variable—type of feedback. Model 2 is superior to Model 1 in terms of the overall significance level (chi-square statistic of 10.489 is significant at 0.005 level). The results show that the likelihood of correct verbal reasoning amongst average students increases when a "single-try" feedback is given during the experimentation.

Table 2. Logistic Regression results for average students
Dependent Variable: VERBAL REASONING

	Model 1		Model 2	
Variable	coefficient	t-ratio	coefficient	t-ratio
CONSTANT	-1.322	-2.349	-2.934	-2.918
TIMING OF FEEDBACK	1.204*	1.619	1.860**	1.986
TYPE OF FEEDBACK			2.335***	2.449
Model Chi-square	2.764[1]		10.489[2]	
% correct prediction	67%		67%	
Asterisk indicates statistical significance: * at 0.10 level; ** at 0.05 level; and *** at 0.01 level; [df]				

The results of logistic regression for average students also indicate that TIMING OF FEEDBACK increases the likelihood of success in GRAPH (a positive coefficient of 1.558). The t-ratio is 1.687 and chi-square of 3.163 is significant at 0.075 level. This effect has a low p-value and it may or may not be a real effect.

The results of logistic regression for failing students show that the likelihood of correct verbal reasoning increases with feedback given during the experimentation (a positive coefficient of 1.649). The t-ratio 2.271 (p-level 0.023) and chi-square statistic 5.611 indicate the model is significant at 0.018 level. The addition of the other explanatory variable (TYPE OF FEEDBACK) did not improve the model. No effect on likelihood of success in GRAPH was shown to be due to either of the two independent variables. Note that in these models, the interactions effects are not present. In fact, they were initially introduced in the model and subsequently eliminated when found insignificant. The present additive models are simpler and easier to interpret.

Discussion

Feedback is recognized as an important element of learning and in this research we have found that feedback plays an important role in the performance of average or failing students of Calculus. Neither timing nor type of feedback had an effect on the learning of high performing students. Thus, our ability to draw any conclusion from the performance of high achieving students is limited. For example, we may speculate that they proceeded through simulations quickly and thus did not experience "timed" feedback. It would be interesting to collect data that might explain "no effect" among high achievers.

The results show that average students improved their verbal reasoning particularly when the feedback was a "single-try-feedback". This result contributes to the debate among feedback researchers whether "multiple-try-feedback" is effective. It contradicts results obtained by Kramarski and Zeichner[16] who found that elaborative "multiple-try-feedback" was effective. One possible explanation may be that the number of tries in this experiment was limited to two. The second possible explanation may have to do with the fact that the elaborative feedback provided on the first try attempted to anticipate and clarify errors in reasoning. The example given earlier in this paper shows that the elaborative feedback used involved long and complex verbal reasoning. It is conceivable that such feedback confused certain students because they did not understand the explanation of their reasoning error. Alternatively, it is also possible that our feedback did not make sense to them because it did not address the reasoning error that they made. It would be interesting to investigate these possibilities in the future by allowing more than two tries and by experimenting with the format of feedback. One particular option of interest might be to have a simple animation accompany the text of the elaborative feedback. We note that since the number of tries was limited to two, it does not seem to be probable that the reason for the preference for a single-try-feedback is the frustration with repeated tries as Clariana [14] suggests.

The likelihood of success in verbal reasoning increased for both the average and failing students when feedback was provided during learning. This is consistent with cognitive theories as well as with the point of view of self-regulation. This kind of feedback focusses on the process of learning and informs students when the learning goal has not been reached. This leads self-regulating students to take corrective measures to close the gap between the current state of knowledge and the goal state according to our model of effective feedback [17]. Unfortunately, the frequency of feedback provides an alternative explanation. Students who received feedback during experimentation had more feedback than those who received feedback only on the formative quiz. It would be interesting to design an experiment in the laboratory and test which of these two alternative explanations is more likely.

The likelihood of success in graphing among average students increases when the feedback is provided during experimentation. The frequency of instances of perfect graphs amongst average students (15 out of 36 graphs) is nearly the same as the frequency of perfect graphs amongst high achievers (17 out of 36). This result is surprising given the 14% gap between the median grades of these two groups of students. On the other hand, the feedback during the experimentation with simulations did not significantly increase the likelihood of success amongst failing students. We found many cases of students who had perfect verbal scores with graphs that had errors. For example, they often reasoned that the slope of a graph at a stationary point should be zero but would fail to draw a zero slope graph at such a point. Two particular concepts, that of inflection point and that of stationary point, were the ones where it most frequently happened that the student would correctly identify an instance in verbal arguments, but make an error at the corresponding point on the graph. This may indicate that student understanding of these concepts is weak. But it may also be that these are students who, while strong in verbal skills are unable to reason in spatial terms. Consequently, their ability to process graphical information may be less developed than their ability to argue and reason verbally. (They just don't **see** it.) Lastly, we may attribute this result to students' epistemological beliefs. They may dismiss the inconsistency believing that verbal and graphical perspectives can lead to mutually inconsistent results; or they may not feel the need to search for and resolve the dissonance between data generated by the two perspectives.

Conclusion

Success in mathematics is the gateway to careers in the sciences, and increasingly in other fields such as economics, social sciences and commerce. Graduates of Calculus courses are expected to use their knowledge to solve problems. This research shows that by providing elaborative feedback during student experimentation with simulations many weaker students acquire skills that experts have. It requires further research to explain how and why this is happening. On the other hand, mathematics instructors may already benefit, having an indication that "single-try-feedback" provided during simulations improves verbal reasoning and to some extent graphical reasoning. Since the web abounds with simulations freely available to instructors, one may hope that students will soon have the opportunity to more easily learn the skills that they need.

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Calculus

and

Computer Supported Learning

Final Report

Appendix 1

Student Consent Form

Student Consent Form

Learning Calculus

Directions to the Student

A team of Mathematics and Science instructors at Champlain College, Dawson College and Vanier College is doing research to investigate the effectiveness of different learning environments for Calculus. Your Calculus instructor has agreed to allow this team to gather information through questionnaires and selected questions/solutions on tests, and if your class uses computers, through recorded log files. This study is being done in collaboration with members of the Centre for the Study of Learning and Performance at Concordia University. All data from this study will be kept **strictly confidential**. This data, and your decision to assist in this effort (or not), will in **no way** influence your grade in this or any other course.

If you are interested in more information, or the results of this research, please contact Helena Dedic, principal investigator, at the Vanier College Science Centre, E512, 744-7016.

I, the undersigned, consent to participate with the assurance that the data will be kept confidential and that they in no way affect my grade in this or any other course. I understand that I have the right to refuse to participate at any time, and that such refusal will also in no way affect my grade in this or any other course. Further I understand that should I decide to participate at this time, I can subsequently change my mind by sending an e-mail to Helena Dedic at dedich@vaniercollege.qc.ca informing her of my decision. In such a circumstance, all data that I have contributed will be withdrawn and my decision will also in no way affect my grade in this or any other course.

PRINT NAME: ______________________________

STUDENT #: ______________

SIGNATURE: ______________________________

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Appendix 2

Student Questionnaires

This survey was designed by a research team at Champlain, Dawson and Vanier College. It is intended to identify factors that affect how people learn mathematics.

Please mark your answers on this questionnaire using **pen or dark pencil**. Make **only** one mark per item, with the exception of item #11 in the **DEMOGRAPHIC INFORMATION** section.

Remember that there are no right or wrong answers to these questions. Your answers should reflect what you actually and honestly think.

Thank you for your cooperation.

DEMOGRAPHIC INFORMATION

1. Gender (circle one). Male Female

2. What year did you graduate from high school? ____________

3. What was the language of instruction in mathematics and sciences at your high school?

   English only French only Other ____________________

4. What is your mother tongue? English French Other ____________

5. How many hours per week do you work for pay? ____________

6. How many courses are you taking this term? ____________

7. Outside of the classroom, how many hours a week do you study mathematics? ____________

8. Do you have a computer at home? Yes No

9. Do you have access to the Internet at home? Yes No

10. How often do you use computers? Circle the most appropriate answer.

    More than once a day Once a day More than once a week Once a week Once a month or less

11. What do you use the computer for? Circle <u>all</u> appropriate answers.

    E-mail Word processing Internet search Learning about interesting subjects Graphing

    Spreadsheet Games Programming Downloading music Chatting with friends

    Other (specify): ______________________________________

12. Rate your computer competence on a scale from 1 (no expertise) to 10 (expert): ____________

1. I experience a "rush", an "AHA!" feeling when I finally get a new math concept.
a) always b) usually c) sometimes d) rarely e) never

2. When it comes to math assignments, I prefer to
a) work completely by myself.
b) work mostly by myself with an occasional consultation with other students.
c) work by myself but I consult frequently with other students.
d) work mostly with other students, although I still like to do some parts by myself.
e) do the whole thing with a group of students.

3. I get easily discouraged.
a) almost never b) rarely c) sometimes d) quite often e) most of the time

4. Heredity determines most of a person's personality.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

5. I prefer a mathematics course that
a) requires me to do only simpler tasks.
b) requires me to do only simpler or intermediate tasks.
c) challenges me without pushing my limits.
d) pushes my limits quite a bit.
e) really pushes my limits.

6. I will use the ideas that I learn in math in other courses.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

7. I dislike math.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

8. I can do even the most difficult problems in the math textbook.
a) always b) usually c) sometimes d) rarely e) never

9. I learn best
a) when I study alone.
b) when I can first discuss a few things and then study alone.
c) when I divide my time evenly between studying with friends and working alone.
d) when I study with my friends and only do review alone.
e) when I study with my friends.

10. I prefer classes in which lectures are interrupted with hands-on activities where I am in control.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

11. Chance has a lot to do with being successful.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

12. Use of computers in courses makes classes more interesting.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

13. What is your opinion about solving math problems?
a) In my opinion it is all about getting the answer.
b) In my opinion it is mostly about getting the answer.
c) In my opinion it is mostly about understanding the ideas used in the solution of the problem.
d) In my opinion it is all about understanding the ideas used in the solution of the problem.
e) I can't decide what I think.

14. When I am stressed, my mind goes blank.
a) almost never b) rarely c) sometimes d) quite often e) most of the time

15. I get easily frustrated when studying math.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

16. Mastery of basic math concepts is a prerequisite for my future studies.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

17. If doing the hard problems did NOT guarantee me a good grade,
a) I would avoid them like the plague.
b) I would stick with the easier exercises.
c) sometimes I would try them but I would not persevere to the end.
d) I would always try them but I would not persevere to the end.
e) I would do them anyway.

18. I'm think I have a good knowledge of basic concepts in math.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

19. I resent revealing good ideas to other people in my group.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

20. I learn best when
a) I explore on my own without the help of a teacher.
b) I explore on my own with a teacher around to help when I need it.
c) I learn the material from a teacher, then explore on my own.
d) I learn the material from a teacher, then explore with a teacher around to help when I need it.
e) a teacher explains everything to me.

21. I am uncomfortable with the idea of using computers to learn.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

22. Successful math students understand the material quickly.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

23. Whatever plans I make, there is always something that will mess them up.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

24. I am most satisfied when a math course
a) requires me to gain deep understanding of all of the concepts covered.
b) requires me to gain deep understanding of most of the concepts covered.
c) requires me to gain deep understanding of some of the concepts covered.
d) requires me to get a good understanding of most of the concepts without going too deep.
e) requires me to get only a very basic understanding of the concepts.

25. When the situation changes, I adjust my plans.
a) almost never b) rarely c) sometimes d) quite often e) most of the time

26. I think that knowledge of math is essential for my future success.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

27. I am unsure that my grades in math courses will be good.
a) always b) usually c) sometimes d) rarely e) never

28. I find math intellectually stimulating.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

29. In a typical group setting, I feel left out.
a) always b) usually c) sometimes d) rarely e) never

30. I enjoy having tasks where
a) I decide by myself how to proceed.
b) I make a plan of how to proceed, and then check it with the teacher before carrying it out.
c) the teacher outlines how to proceed and I provide the details.
d) the teacher provides step-by-step instructions.
e) all I have to do is fill in the blanks.

31. If I could I would avoid enrolling in a course in which I have to use computers.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

32. I trust my judgement.
a) almost never b) rarely c) sometimes d) quite often e) most of the time

33. Being at the right place at the right time is essential for getting what you want in life.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

34. Knowledge of mathematics
a) depends entirely on the amount of effort one puts in to learning it.
b) depends mostly on the effort one puts in to learning it.
c) depends equally on effort and talent for mathematics.
d) depends mostly on one's talent for mathematics.
e) depends entirely on one's talent for mathematics.

35. I get angry when I am faced with challenging math problems.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

36. In my everyday experience, I will use the logical thinking that I learned in math courses.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

37. When I don't understand ideas presented in mathematics courses,
a) it doesn't bother me at all. I only care about my grades.
b) it bothers me a little but if my grades are already good I will not try to fix it.
c) it bothers me a lot but if my grades are already good I will not try to fix it.
d) it bothers me a lot. Even if my grades are already good I will try to fix it.
e) it bothers me a lot. Even if my grades are already good I will not stop until I have fixed it.

38. I expect to understand even the most complex ideas presented by math teachers.
a) always b) usually c) sometimes d) rarely e) never

39. I need a pat on the shoulder in order to know that I have done well.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

40. I like discussing ideas and solutions with other people.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

41. Teachers' objectives in solving math problems in class should be
    a) entirely to demonstrate and to drill students' skills.
    b) mainly to demonstrate and to drill students' skills.
    c) mainly to enhance students' understanding of the theory.
    d) entirely to enhance students' understanding of the theory.
    e) I can't decide what I think.

42. Computers make communication with my teachers and classmates easier.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

43. Intelligence is a given and cannot be trained or repressed.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

44. When I don't understand something in a math course,
    a) I always keep working until I understand the concepts.
    b) occasionally I stop working before I understand the concepts and I memorize the formulas instead.
    c) I often stop working before I understand the concepts and I memorize the formulas instead.
    d) very often I stop working before I understand the concepts and I memorize the formulas instead.
    e) I memorize the formulas and leave it at that.

45. I know where to find the information that I need.
    a) almost never b) rarely c) sometimes d) quite often e) most of the time

46. I can succeed in math.
    a) always b) usually c) sometimes d) rarely e) never

47. Math is boring.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

48. It is useful to work on math assignments in a group because we can help each other.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

49. The skills I learn in mathematics courses are useless in everyday life activities.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

50. When faced with a difficult problem in math I prefer to rely on my own resources to find the solution.
    a) very characteristic of me
    b) rather characteristic of me
    c) somewhat characteristic of me
    d) rather uncharacteristic of me
    e) very uncharacteristic of me

51. Using computers to learn math is a waste of time.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

52. Making several unsuccessful attempts when solving math problems
    a) is perfectly natural.
    b) is relatively normal.
    c) indicates a potential problem with student's ability to learn math.
    d) indicates that a student has a problem when it comes to math.
    e) a clear sign of a student who is bad in math.

53. If I successfully accomplish my task, it's because it was an easy one
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

54. When a situation requires a change of plan or strategy, I feel confused or anxious.
    a) almost never b) rarely c) sometimes d) quite often e) most of the time

55. I avoid taking optional math classes.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

56. When I am curious about an idea I
    a) like to learn it even if it's very difficult to understand.
    b) like to learn it even if it is somewhat difficult to understand.
    c) like to learn it only if it's reasonably easy to understand.
    d) like to learn it only if it's very easy to understand.
    e) forget about it quickly and don't attempt to gain understanding.

57. I expect to be one of the weak students in math classes.
    a) always b) usually c) sometimes d) rarely e) never

58. Doing mathematics gives me satisfaction.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

59. In a typical group setting, I feel intimidated by my more competent group-mates.
    a) always b) usually c) sometimes d) rarely e) never

60. Math is
    a) all about understanding general ideas.
    b) mostly about understanding general ideas.
    c) evenly divided between understanding general ideas and carrying out procedures step-by-step.
    d) mostly about carrying out procedures step-by-step.
    e) all about carrying out procedures step-by-step.

61. I get anxious when I don't get step-by-step instructions on how to accomplish a task.
    a) very characteristic of me
    b) rather characteristic of me
    c) somewhat characteristic of me
    d) rather uncharacteristic of me
    e) very uncharacteristic of me

62. Computer-based instruction should be included in college level courses.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

63. You cannot cheat your fate.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

64. When presented with a math problem that I am not sure I will be able to finish,
    a) I work on it until it is solved, no matter what.
    b) I give it my best shot anyways, but move on eventually if it doesn't work out.
    c) I try to solve it, but as soon as I get stuck I abandon the attempt.
    d) I try to solve it only if the teacher forces me to.
    e) I don't even try the problem.

65. When something I want doesn't work out, I rapidly get back on my feet.
    a) almost never b) rarely c) sometimes d) quite often e) most of the time

66. When in a group setting, I feel comfortable speaking my mind.
    a) always b) usually c) sometimes d) rarely e) never

67. School success is mostly a result of one's socio-economic background.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

68. My solutions for math problems are correct.
    a) always b) usually c) sometimes d) rarely e) never

69. I think that understanding concepts in mathematics is only useful to mathematicians or people working in related fields.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

70. Math is learned slowly by solving problems, learning from mistakes and figuring out the meaning of ideas.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

71. When I come across a difficult problem in mathematics I immediately seek my instructor's help.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

72. Use of computers makes learning math easier.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

73. People are lonely because they are not given the chance to meet new people.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

74. When I fail, I am devastated for a long time.
a) almost never b) rarely c) sometimes d) quite often e) most of the time

75. When I come across problems for which I CANNOT find the correct answers in the text,
a) I will ignore them.
b) I will read them, but will not really attempt to solve them.
c) I will try to do them, but if I get stuck I will stop immediately.
d) I will do them anyway, but I will be unsure about my solution unless the teacher checks it.
e) I will do them anyway. I can tell by myself if I have essentially solved a problem.

76. Math is one of my favourite subjects.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

77. By trying too hard to understand ideas in math, I end up being more confused.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

78. Working in a group motivates me to spend more time and energy on my assignments.
a) always b) usually c) sometimes d) rarely e) never

79. Compared to other subjects, I write math exams
a) much less confidently than I write the others.
b) less confidently than I write the others.
c) as confidently as I write the others.
d) more confidently than I write the others.
e) much more confidently than I write the others.

80. It is comforting to me that math problems usually have just one right answer.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

81. Computers are only good for fast numerical computations.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

82. Working in a group makes even boring tasks enjoyable.
a) always b) usually c) sometimes d) rarely e) never

83. I accept my mistakes as a learning opportunity.
a) almost never b) rarely c) sometimes d) quite often e) most of the time

84. When presented with optional exercises in addition to assigned ones,
a) I will do both sets of problems.
b) I will solve the assigned exercises and I will attempt to do the optional ones.
c) I will solve the assigned exercises and I will glance over the optional ones.
d) I will do only the assigned exercises.
e) I will do neither.

85. To be sure of myself I need to interact with my teacher.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

86. To be good in math a student needs to
a) recall solutions of problems seen in class or the text.
b) apply solutions of problems seen in class or the text, but to slightly different ones.
c) modify solutions of problems seen in class or the text and then apply them to new ones.
d) combine solutions of problems seen in class or the text with ideas learned in the course to solve new problems.
e) apply ideas learned in the course to solve new problems.

87. I have fun solving hard problems in math.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

88. If there were no grades to evaluate my success, I would
a) have no idea of how I did or whether I learned the material.
b) have only a vague idea of how I did and not be sure about how much I have learned.
c) have some doubts about how I did and how much I have learned.
d) have a pretty good idea of how I did and of how much I learned.
e) know exactly how I did and how much I learned.

89. If you set realistic goals, you can succeed no matter what.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

90. I am able to apply what I have learned to new situations.
a) almost never b) rarely c) sometimes d) quite often e) most of the time

This survey was designed by a research team at Champlain, Dawson and Vanier College. It is intended to identify factors that affect how people learn mathematics.

Please mark your answers on this questionnaire using **pen or dark pencil**. Make **only** one mark per item, with the exception of item #9 in the **COURSE INFORMATION** section.

Remember that there are no right or wrong answers to these questions. Your answers should reflect what you actually and honestly think.

Thank you for your cooperation.

COURSE INFORMATION

1. Name: ______________________ Student Number: ______________
2. Outside of the classroom, how many hours a week do you study mathematics? __________

3. When you compare the workload in this Calculus course to the workload in your other science courses, do you consider it to be
a) very heavy? b) heavy? c) average? d) lighter? e) very light?

4. At the end of a lesson I walk away from the classroom feeling that
a) I understand the idea.
b) I understand the basics.
c) I understand some ideas but not all of them.
d) I understand just a few ideas.
e) I am very confused.

5. I attended in this course
a) more than 90% of the classes.
b) over 80% of the classes.
c) over 70% of the classes.
d) more than half of the classes.
e) less than half of the classes.

6. The instruction in this course was
a) very good.
b) good.
c) satisfactory.
d) fair.
e) unsatisfactory.

7. Do you have a computer at home? Yes No

8. Do you have access to the Internet at home? Yes No

9. How often do you use computers? Circle the most appropriate answer.

More than once a day Once a day More than once a week Once a week Once a month or less

10. What do you use the computer for? Circle **<u>all</u>** appropriate answers.

E-mail Word processing Internet search Learning about interesting subjects Graphing

Spreadsheet Games Downloading music Chatting with friends School work

Other (specify): ______________________________

11. Rate your computer competence on a scale from 1 (no expertise) to 10 (expert): __________

1. I experience a "rush", an "AHA!" feeling when I finally get a new math concept.
   a) always b) usually c) sometimes d) rarely e) never

2. When it comes to math assignments, I prefer to
   a) work completely by myself.
   b) work mostly by myself with an occasional consultation with other students.
   c) work by myself but I consult frequently with other students.
   d) work mostly with other students, although I still like to do some parts by myself.
   e) do the whole thing with a group of students.

3. I get easily discouraged.
   a) almost never b) rarely c) sometimes d) quite often e) most of the time

4. Heredity determines most of a person's personality.
   a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

5. I prefer a mathematics course that
   a) requires me to do only simpler tasks.
   b) requires me to do only simpler or intermediate tasks.
   c) challenges me without pushing my limits.
   d) pushes my limits quite a bit.
   e) really pushes my limits.

6. I will use the ideas that I learn in math in other courses.
   a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

7. I dislike math.
   a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

8. I can do even the most difficult problems in the math textbook.
   a) always b) usually c) sometimes d) rarely e) never

9. I learn best
   a) when I study alone.
   b) when I can first discuss a few things and then study alone.
   c) when I divide my time evenly between studying with friends and working alone.
   d) when I study with my friends and only do review alone.
   e) when I study with my friends.

10. I prefer classes in which lectures are interrupted with hands-on activities where I am in control.
    a) very characteristic of me
    b) rather characteristic of me
    c) somewhat characteristic of me
    d) rather uncharacteristic of me
    e) very uncharacteristic of me

11. Chance has a lot to do with being successful.
    a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

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13. What is your opinion about solving math problems?
    a) In my opinion it is all about getting the answer.
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    c) In my opinion it is mostly about understanding the ideas used in the solution of the problem.
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    a) almost never b) rarely c) sometimes d) quite often e) most of the time

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a) I would avoid them like the plague.
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c) sometimes I would try them but I would not persevere to the end.
d) I would always try them but I would not persevere to the end.
e) I would do them anyway.

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a) very characteristic of me
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19. I resent revealing good ideas to other people in my group.
a) very characteristic of me
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c) I learn the material from a teacher, then explore on my own.
d) I learn the material from a teacher, then explore with a teacher around to help when I need it.
e) a teacher explains everything to me.

21. I am uncomfortable with the idea of using computers to learn.
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a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

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d) the teacher provides step-by-step instructions.
e) all I have to do is fill in the blanks.

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    c) I often stop working before I understand the concepts and I memorize the formulas instead.
    d) very often I stop working before I understand the concepts and I memorize the formulas instead.
    e) I memorize the formulas and leave it at that.

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    a) almost never b) rarely c) sometimes d) quite often e) most of the time

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a) always b) usually c) sometimes d) rarely e) never

60. Math is
a) all about understanding general ideas.
b) mostly about understanding general ideas.
c) evenly divided between understanding general ideas and carrying out procedures step-by-step.
d) mostly about carrying out procedures step-by-step.
e) all about carrying out procedures step-by-step.

61. I get anxious when I don't get step-by-step instructions on how to accomplish a task.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

62. Computer-based instruction should be included in college level courses.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

63. You cannot cheat your fate.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

64. When presented with a math problem that I am not sure I will be able to finish,
a) I work on it until it is solved, no matter what.
b) I give it my best shot anyways, but move on eventually if it doesn't work out.
c) I try to solve it, but as soon as I get stuck I abandon the attempt.
d) I try to solve it only if the teacher forces me to.
e) I don't even try the problem.

65. When something I want doesn't work out, I rapidly get back on my feet.
a) almost never b) rarely c) sometimes d) quite often e) most of the time

66. When in a group setting, I feel comfortable speaking my mind.
a) always b) usually c) sometimes d) rarely e) never

67. School success is mostly a result of one's socio-economic background.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

68. My solutions for math problems are correct.
a) always b) usually c) sometimes d) rarely e) never

69. I think that understanding concepts in mathematics is only useful to mathematicians or people working in related fields.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

70. Math is learned slowly by solving problems, learning from mistakes and figuring out the meaning of ideas.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

71. When I come across a difficult problem in mathematics I immediately seek my instructor's help.
a) very characteristic of me
b) rather characteristic of me
c) somewhat characteristic of me
d) rather uncharacteristic of me
e) very uncharacteristic of me

72. Use of computers makes learning math easier.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

73. People are lonely because they are not given the chance to meet new people.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

74. When I fail, I am devastated for a long time.
a) almost never b) rarely c) sometimes d) quite often e) most of the time

75. When I come across problems for which I CANNOT find the correct answers in the text,
a) I will ignore them.
b) I will read them, but will not really attempt to solve them.
c) I will try to do them, but if I get stuck I will stop immediately.
d) I will do them anyway, but I will be unsure about my solution unless the teacher checks it.
e) I will do them anyway. I can tell by myself if I have essentially solved a problem.

76. Math is one of my favourite subjects.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

77. By trying too hard to understand ideas in math, I end up being more confused.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

78. Working in a group motivates me to spend more time and energy on my assignments.
a) always b) usually c) sometimes d) rarely e) never

79. Compared to other subjects, I write math exams
a) much less confidently than I write the others.
b) less confidently than I write the others.
c) as confidently as I write the others.
d) more confidently than I write the others.
e) much more confidently than I write the others.

80. It is comforting to me that math problems usually have just one right answer.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

81. Computers are only good for fast numerical computations.
a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

82. Working in a group makes even boring tasks enjoyable.
a) always b) usually c) sometimes d) rarely e) never

83. I accept my mistakes as a learning opportunity.
a) almost never b) rarely c) sometimes d) quite often e) most of the time

84. When presented with optional exercises in addition to assigned ones,
a) I will do both sets of problems.
b) I will solve the assigned exercises and I will attempt to do the optional ones.
c) I will solve the assigned exercises and I will glance over the optional ones.
d) I will do only the assigned exercises.
e) I will do neither.

85. To be sure of myself I need to interact with my teacher.
   a) very characteristic of me
   b) rather characteristic of me
   c) somewhat characteristic of me
   d) rather uncharacteristic of me
   e) very uncharacteristic of me

86. To be good in math a student needs to
   a) recall solutions of problems seen in class or the text.
   b) apply solutions of problems seen in class or the text, but to slightly different ones.
   c) modify solutions of problems seen in class or the text and then apply them to new ones.
   d) combine solutions of problems seen in class or the text with ideas learned in the course to solve new problems.
   e) apply ideas learned in the course to solve new problems.

87. I have fun solving hard problems in math.
   a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

88. If there were no grades to evaluate my success, I would
   a) have no idea of how I did or whether I learned the material.
   b) have only a vague idea of how I did and not be sure about how much I have learned.
   c) have some doubts about how I did and how much I have learned.
   d) have a pretty good idea of how I did and of how much I learned.
   e) know exactly how I did and how much I learned.

89. If you set realistic goals, you can succeed no matter what.
   a) strongly agree b) agree c) neither agree nor disagree d) disagree e) strongly disagree

90. I am able to apply what I have learned to new situations.
   a) almost never b) rarely c) sometimes d) quite often e) most of the time

Calculus

and

Computer Supported Learning

Final Report

Appendix 3

Coding and Scoring Schema

L=Limit C=Continuity D=Differentiation G=Graphing A=Application of Differential Calculus	Co=Conceptual Al=Algorithmic	LD1=Level of Difficulty is Simple LD2=Level of Difficulty is Moderate LD3=Level of Difficulty is Complex

Problem P3
L, Al, a)=LD2, b)=LD2, c)=LD3, d)=LD3
On limits at ± infinity / infinite limits

1. Determine each of the following limits, showing all work:

   a) $\lim_{x\to-\infty} \frac{x^3+4x}{2x^3-x^2+5}$

Solution:

We note that this is a limit of a rational function. Further, x approaches $-\infty$ means we are examining behaviour at the left edge of the graph. We know that at either edge, a polynomial is dominated by its leading term, *i.e.*, its behaviour is entirely determined by the leading term. Thus, the behaviour of a rational function will be determined by the ratio of its leading terms.

$$\lim_{x\to-\infty} \frac{x^3+4x}{2x^3-x^2+5} = \lim_{x\to-\infty} \frac{x^3}{2x^3} = \lim_{x\to-\infty} \frac{1}{2} = \frac{1}{2}$$

This limit tells us that at the left edge of the graph the given rational function will be asymptotic to the horizontal line, $y = ½$.

Coding:

1. Copying the problem
   1. Correctly — code = 0
   2. Incorrectly — code = 1
   3. No real attempt to solve problem — code = 2

Remaining numbers are only coded if 1. above is coded as 0.

2. Count the number of steps used in the solution: count a step for each new expression on a new line, or after = or after ⇒ , but don't count copying of the original problem as a step.

3. Count the number of times the symbol $\lim_{x\to\infty}$ is omitted when it shouldn't be, *e.g.*, $\lim_{x\to-\infty} \frac{x^3+4x}{2x^3-x^2+5} = \frac{x^3}{2x^3}$.

4. Count the number of times the symbol $\lim_{x\to\infty}$ is written when it shouldn't be, *e.g.*,

$$\lim_{x\to-\infty} \frac{\frac{x^3}{x^3}+\frac{4x}{x^3}}{\frac{2x^3}{x^3}-\frac{x^2}{x^3}+\frac{5}{x^3}} = \lim_{x\to-\infty} \frac{1+0}{2-0+0}$$

5. Count the number of times the symbol $\lim_{x\to\infty}$ is written without an expression of any type, *e.g.*, $\lim_{x\to\infty} = \ldots$.

6. Count the number of times the symbol lim is written without writing $x\to\infty$ underneath.

7. Code the method of solution:
   1. If both numerator and denominator are divided by the highest power of x existing in the denominator, or if the expression is converted to a ratio of the leading terms from both the numerator and denominator. — code = 0
   2. Anything else — code = 1

8. Code solution as:

1. Correct code = 0
2. Correct except for trivial arithmetic error code = 1
3. Correct, given a small algebraic error code = 2
4. Incorrect code = 3

9. Code for use of the symbol ∞ in the computation of the answer:
   1. Not using it in the computation (we allow use of a calculation leading to an expression "$\frac{\infty}{\infty}$" in a statement that justifies the subsequent use of some valid method, or use of a valid method ending in ratio of # and ∞) code = 0
   2. Using ∞ in the computation of the limit code = 1

10. Code student discussion:
    1. Nothing written is incorrect. code = 0
    2. Something written is incorrect. code = 1
    3. Nothing is written. code = 2

b) $\lim_{x\to -3}\frac{x^2+4x+3}{x+3}$

Solution:

We note that this is a limit of a rational function. Rational functions are continuous where they are defined, hence we begin by hoping for continuity at x = -3 and simply substituting in.

$$\lim_{x\to -3}\frac{x^2+4x+3}{x+3} = "\frac{(-3)^2+4(-3)+3}{(-3)+3}" = "\frac{9-12+3}{-3+3}" = "\frac{0}{0}"$$

While our first attempt did not work, it tells us that the rational function has a discontinuity at x = -3, but that more algebra will be required if we are to determine whether this is a removable discontinuity (graph has a missing point) or an infinite discontinuity (graph has a vertical asymptote). We know that a polynomial has a zero at x = -3 if and only if (x - (-3)) = (x + 3) is a factor, thus the zeroes of both the numerator and denominator must be due to this factor. We attempt to cancel this factor from both numerator and denominator, and then hope that the resulting rational function will be continuous so that the limit can be computed by substitution.

$$\lim_{x\to -3}\frac{x^2+4x+3}{x+3} = \lim_{x\to -3}\frac{(x+3)(x+1)}{x+3} = \lim_{x\to -3} x+1 = -3+1 = -2$$

Note that since this limit exists, we now know that the given rational function has a removable discontinuity at x = -3 which will appear on a graph as a missing point at (-3,-2).

Coding:

1. Copying the problem
   1. Correctly code = 0
   2. Incorrectly code = 1
   3. No real attempt to solve problem code = 2

Remaining numbers are only coded if 1. above is coded as 0.

2. Count the number of steps used in the solution: count a step for each new expression on a new line, or after = or after ⇔, but don't count copying of the original problem as a step.

3. Count the number of times the symbol $\lim_{x\to -3}$ is omitted when it shouldn't be, *e.g.*, $\lim_{x\to -3}\frac{x^2+4x+3}{x+3} = x+1$.

4. Count the number of times the symbol $\lim_{x\to -3}$ is written when it shouldn't be, *e.g.*, $\lim_{x\to -3} x+1 = \lim_{x\to -3} -3+1$

5. Count the number of times the symbol $\lim\limits_{x\to -3}$ is written without an expression of any type, *e.g.*, $\lim\limits_{x\to -3} = \ldots$.

6. Count the number of times the symbol lim is written without writing $x\to -3$ underneath.

7. Code the method of solution:
   1. If any of these methods are used: factoring, or synthetic division, or trial and error — code = 0
   2. Anything else — code = 1

8. Code solution as:
   1. Correct — code = 0
   2. Correct except for trivial arithmetic error — code = 1
   3. Correct, given a small algebraic error — code = 2
   4. Incorrect — code = 3

9. Code algebraic error:
   1. No error in algebra, *i.e.*, obtain $x + 1$ en route to solution — code = 0
   2. Solution indicates that an attempt is made to divide both the numerator and the denominator by $(x + 3)$ but the result is different from $x + 1$ — code = 1
   3. No attempt at division by $(x + 3)$ is made — code = 2

10. Code student discussion:
    1. Nothing written is incorrect. — code = 0
    2. Something written is incorrect. — code = 1
    3. Nothing is written. — code = 2

c) $\lim\limits_{x\to -\infty} \dfrac{4x-3}{\sqrt{x^2+x-5}}$

Solution:

We note that this is a limit of an algebraic function. Further, the limit is asking us to determine the behaviour of the function at the left edge of the graph. Reasoning in similar fashion to such limits for polynomial and rational functions, we know that only the highest power terms of both numerator and denominator will affect this limit so we simplify the function. We also note that $\sqrt{x^2} = |x|$, and at the left edge, $|x| = -x$. Using this information we can solve this limit.

$$\lim_{x\to -\infty} \frac{4x-3}{\sqrt{x^2+x-5}} = \lim_{x\to -\infty} \frac{4x}{\sqrt{x^2}} = \lim_{x\to -\infty} \frac{4x}{|x|} = \lim_{x\to -\infty} \frac{4\cancel{x}}{-\cancel{x}} = \lim_{x\to -\infty} -4 = -4$$

The result of this limit calculation tells us that at the left edge the given function is asymptotic to the horizontal line $y = -4$.

Coding:

Remaining numbers are only coded if 1. above is coded as 0.

1. Copying the problem
   1. Correctly — code = 0
   2. Incorrectly — code = 1
   3. No real attempt to solve problem — code = 2

Remaining numbers are only coded if 1. above is coded as 0.

2. Count the number of steps used in the solution: count a step for each new expression on a new line, or after = or after ⇔ , but don't count copying of the original problem as a step.

3. Count the number of times the symbol $\lim\limits_{x\to-\infty}$ is omitted when it shouldn't be, *e.g.*, $\lim\limits_{x\to-\infty}\frac{4x-3}{\sqrt{x^2+x-5}}=\frac{4x}{\sqrt{x^2}}$.

4. Count the number of times the symbol is written when it shouldn't be, *e.g.*,

$$\lim_{x\to-\infty}\frac{\frac{4x}{x}-\frac{3}{x}}{-\sqrt{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{5}{x^2}}}=\lim_{x\to-\infty}\frac{4-0}{-\sqrt{1+0-0}}$$

5. Count the number of times the symbol $\lim\limits_{x\to-\infty}$ is written without an expression of any type, e.g., .

6. Count the number of times the symbol lim is written without writing $x\to-\infty$

7. Code the method of solution:
   1. If any of these methods are used: divide by the highest power in the denominator (x or x^2) or the ratio of the leading terms or rationalize and then use either division by the highest power of the denominator or the ratio of leading terms — code = 0
   2. Anything else — code = 1

8. Code solution as:
   1. Correct — code = 0
   2. Correct except for trivial arithmetic error — code = 1
   3. Correct, given a small algebraic error — code = 2
   4. Incorrect — code = 3

9. Code for algebraic error (if symbols are all replaced there is no algebra, hence no algebra errors):
   1. No error in algebra — code = 0
   2. Algebra notational error, but student still works past it correctly. — code = 1
   3. The only algebraic error is $\sqrt{x^2}=x$ — code = 2
   4. Any number of algebraic errors, other than $\sqrt{x^2}=x$ — code = 3
   5. Any number of algebraic errors, one of which is $\sqrt{x^2}=x$ — code = 4

10. Code for use of the symbol ∞ in the computation of the answer:
    1. Not using it in the computation (we allow use of a calculation to obtain an expression "$\frac{\infty}{\infty}$" in a statement that justifies the subsequent use of some valid method, or use of a valid method ending in ratio of # and ∞) — code = 0
    2. Using ∞ in the computation of the limit — code = 1

11. Code student discussion:
    1. Nothing written is incorrect. — code = 0
    2. Something written is incorrect. — code = 1
    3. Nothing is written. — code = 2

d) $\lim\limits_{x\to\infty}\left(\sqrt{x^2-5x+7}-2x\right)$

Solution:

We note that this is a limit of an algebraic function. Further, the limit is asking us to determine the behaviour of the function at the right edge of the graph. Reasoning in similar fashion to such limits for polynomial and rational functions, we know that only the highest power terms of both numerator and denominator will affect this limit, as long as we are looking at a ratio. Thus, we must first rewrite this function as a ratio, and then simplify the function keeping only highest power terms. We also note that $\sqrt{x^2}=|x|$, and at the right edge, $|x|=x$. Using this information, we can solve this limit.

$$\lim_{x\to\infty}\left(\sqrt{x^2-5x+7}-2x\right)=\lim_{x\to\infty}\left(\sqrt{x^2-5x+7}-2x\right)\times\frac{\left(\sqrt{x^2-5x+7}+2x\right)}{\left(\sqrt{x^2-5x+7}+2x\right)}=\lim_{x\to\infty}\frac{\left(\left(\sqrt{x^2-5x+7}\right)^2-(2x)^2\right)}{\left(\sqrt{x^2-5x+7}+2x\right)}$$

$$=\lim_{x\to\infty}\frac{\left(x^2-5x+7-4x^2\right)}{\left(\sqrt{x^2-5x+7}+2x\right)}=\lim_{x\to\infty}\frac{\left(-3x^2-5x+7\right)}{\left(\sqrt{x^2-5x+7}+2x\right)}=\lim_{x\to\infty}\frac{-3x^2}{\left(\sqrt{x^2}+2x\right)}=\lim_{x\to\infty}\frac{-3x^2}{\left(|x|+2x\right)}$$

$$=\lim_{x\to\infty}\frac{-3x^2}{(x+2x)}=\lim_{x\to\infty}\frac{-3x^2}{(3x)}=\lim_{x\to\infty}-x=-\infty$$

The result of this limit calculation tells us that at the right edge a graph of the given function heads downwards towards $-\infty$. Looking more closely we note that this the function behaves like a linear function, $y = -x$, at the right edge, thus we have determined that this function has an oblique asymptote at the right edge.

Coding:

1. Copying the problem
   1. Correctly — code = 0
   2. Incorrectly — code = 1
   3. No real attempt to solve problem — code = 2

Remaining numbers are only coded if 1. above is coded as 0.

2. Count the number of steps used in the solution: count a step for each new expression on a new line, or after = or after $\Rightarrow$, but don't count copying of the original problem as a step.

3. Count the number of times the symbol $\lim\limits_{x\to\infty}$ is omitted when it shouldn't be, *e.g.*,

$$\lim_{x\to\infty}\left(\sqrt{x^2-5x+7}-2x\right)=\left(\sqrt{x^2-5x+7}-2x\right)\times\frac{\left(\sqrt{x^2-5x+7}+2x\right)}{\left(\sqrt{x^2-5x+7}+2x\right)}.$$

4. Count the number of times the symbol $\lim\limits_{x\to\infty}$ is written when it shouldn't be, *e.g.*,

$$\lim_{x\to\infty}\frac{\frac{-3x^2}{x}-\frac{5x}{x}+\frac{7}{x}}{\sqrt{\frac{x^2}{x^2}-\frac{5x}{x^2}+\frac{7}{x^2}}+\frac{2x}{x}}=\lim_{x\to\infty}\frac{-\infty-5+0}{\sqrt{1-0+0}+2}.$$

5. Count the number of times the symbol $\lim\limits_{x\to\infty}$ is written without an expression of any type, *e.g.*, $\lim\limits_{x\to\infty}=\ldots$.

6. Count the number of times the symbol lim is written without writing $x\to\infty$

7. Code the method of solution:
   1. If the first step is to rationalize and then any of these methods are used: division by the highest power of x in the denominator or the ratio of leading terms — code = 0
   2. Anything else — code = 1

8. Code solution as:
   1. Correct — code = 0
   2. Correct except for trivial arithmetic error — code = 1
   3. Correct, given a small algebraic error — code = 2
   4. Incorrect — code = 3

9. Code solution as:
   1. There is evidence that student believes that the square root of a quadratic has the same degree as a linear function when $x\to\infty$, then — code = 0
   2. No evidence of this belief — code = 1

10. Code for use of the symbol ∞ in the computation of the answer:
   1. Not using it in the computation (we allow use of a calculation involving "∞ - ∞" and "∞ / ∞" in a statement that justifies the subsequent use of some valid method, or use of a valid method ending in ratio of # and ∞) code = 0
   2. Using ∞ in the computation of the limit code = 1

11. Code student discussion:
   1. Nothing written is incorrect. code = 0
   2. Something written is incorrect. code = 1
   3. Nothing is written. code = 2

12. Code for algebraic error:
   1. No algebraic errors code = 0
   2. At least one algebraic error code = 1

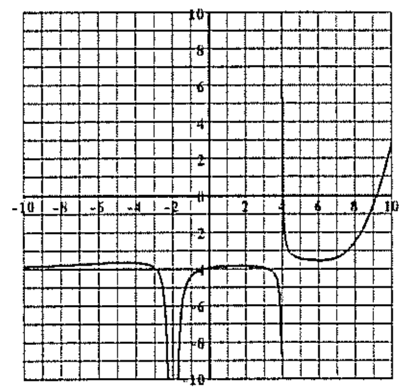
L=Limit C=Continuity D=Differentiation G=Graphing A=Application of Differential Calculus	Co=Conceptual Al=Algorithmic	LD1=Level of Difficulty is Simple LD2=Level of Difficulty is Moderate LD3=Level of Difficulty is Complex

Problem P4
L, Co, LD1
2. For the function g whose graph is given, determine, with reasons, each of the following limits:

a) $\lim_{x \to \infty} g(x)$ b) $\lim_{x \to -\infty} g(x)$

c) $\lim_{x \to -2} g(x)$ d) $\lim_{x \to 4^+} g(x)$

e) provide equations for all asymptotes to $g(x)$:



Solution:
a) We note that at the right edge of the x-axis the graph appears to be heading upwards, getting ever larger, hence towards ∞. Thus, $\lim_{x \to \infty} g(x) = \infty$

b) We note that at the left edge of the x-axis the graph appears to be getting closer and closer to the horizontal line, $y = -4$. Thus, $\lim_{x \to -\infty} g(x) = -4$

c) We note that as x gets closer and closer to -2, from either side of -2, the y-values on the graph appear to be heading downwards, towards -∞. Thus, $\lim_{x \to -2} g(x) = -\infty$.

d) We note that as x gets closer and closer to 4, but from the right side of 4, the y-values on the graph appear to be heading upwards, getting ever larger, towards ∞. Thus, $\lim_{x \to 4^+} g(x) = \infty$

e) At the left edge there is a horizontal asymptote: $y = -4$. There are two vertical asymptotes: $x = -2$ and $x = 4$.

Coding: N.B. For coding purposes ∞ = infinity, U = undefined = DNE = Does Not Exist, and code = 99 when leaving blank

a)
1. Correct answer:
   1. Answer is ∞ or (∞ and DNE) code = 0
   2. Answer is DNE (but not also ∞) code = 1
   3. Answer is anything else code = 2
   4. No answer code = 3

2. If code for 1. above is 1:
   1. Any correct explanation for DNE (*e.g.*, diverges to ∞, approaches ∞, or just not finite, ...) code = 0
   2. No correct explanation for DNE code = 1

3. Discussion (anything beyond just a written answer):
   1. Specific reference to not having HA at right edge but instead increasing towards or diverging towards ∞ code = 0
   2. Correct information (reference to right edge of graph behaviour) code = 1
   3. Incorrect information (*e.g.*, reference to left edge of graph behaviour, or something else) code = 2
   4. No discussion code = 3

b)

4. Correct answer:
   1. Answer is -4 code = 0
   2. Answer is 4 code = 1
   3. Answer is anything else code = 2
   4. No answer code = 3

5. Discussion (anything beyond just a written answer):
   1. Specific reference to having HA at left edge and decreasing towards or coming from above code = 0
   2. Correct information (reference to left edge of graph behaviour) code = 1
   3. Incorrect information (*e.g.*, reference to right edge of graph behaviour, or something else) code = 2
   4. No discussion code = 3

c)

6. Correct answer:
   1. Answer is $-\infty$ or($-\infty$ and DNE) code = 0
   2. Answer is DNE (but not also $-\infty$) code = 1
   3. Answer is anything else code = 2
   4. No answer code = 3

7. If code for 1. above is 1:
   1. Any correct explanation for DNE (*e.g.*, diverges to $-\infty$, approaches $-\infty$, or just not finite, ...) code = 0
   2. No correct explanation for DNE code = 1

8. Discussion (anything beyond just a written answer):
   1. Specific reference to having VA at $x = -2$ and decreasing/diverging towards $-\infty$ on both sides code = 0
   2. Correct information (reference to behaviour as x gets close to -2) code = 1
   3. Incorrect information (*e.g.*, reference to anything else) code = 2
   4. No discussion code = 3

d)

9. Correct answer:
   1. Answer is ∞ or (∞ and DNE) code = 0
   2. Answer is DNE code = 1
   3. Answer is $-\infty$ or ($-\infty$ and DNE) code = 2
   4. Answer is anything else code = 3
   5. No answer code = 4

10. If code for 9. above is 1:
    1. Any correct explanation for DNE (*e.g.*, diverges to ∞, approaches ∞, or just not finite, ...) code = 0
    2. No correct explanation for DNE code = 1

11. If code for 9. above is 2:
    1. Any evidence that the student just misunderstood which side to approach from code = 0
    2. No such evidence code = 1

12. Discussion (anything beyond just a written answer):
    1. Specific reference to having VA at $x = 4$ from the right and increasing/diverging towards ∞ on that side code = 0
    2. Correct information (reference to what happens as x gets close to 4) code = 1
    3. Incorrect information (*e.g.*, reference to anything else) code = 2
    4. No discussion code = 3

e)

For left edge:

13. Depiction of asymptote:
    1. Depict asymptote at left edge via any of $y = -4$, $y = 4$, in words, -4, 4, $x = -4$ or $x = 4$ code = 0
    2. No depiction of asymptote at left edge code = 1

14. If code for 13 above is 0
    1. If answer is $y = -4$ code = 0
    2. If answer is $y = -4$, but in words instead of equation code = 1
    3. If answer is $y = 4$ code = 2
    4. If answer is -4 or 4 code = 3
    5. If answer is $x = -4$ code = 4
    6. If answer is $x = 4$ code = 5

For $x = -2$:

15. Depiction of asymptote:
    1. Depict asymptote at $x = -2$ via any of $x = -2$, $x = 2$, in words, -2, 2, $y = -2$ or $y = 2$ code = 0
    2. No depiction of asymptote at $x = -2$ code = 1

16. If code for 15 above is 0
    1. If answer is $x = -2$ code = 0
    2. If answer is $x = -2$, but in words instead of equation code = 1
    3. If answer is $x = 2$ code = 2
    4. If answer is -2 or 2 code = 3
    5. If answer is $y = -2$ code = 4
    6. If answer is $y = 2$ code = 5

For $x = 4$:

17. Depiction of asymptote:
    1. Depict asymptote at $x = 4$ via any of $x = 4$, $x = -4$, in words, 4, -4, $y = 4$ or $y = -4$ code = 0
    2. No depiction of asymptote at $x = 4$ code = 1

18. If code for 15 above is 0
    1. If answer is $x = 4$ code = 0
    2. If answer is $x = 4$, but in words instead of equation code = 1
    3. If answer is $x = -4$ code = 2
    4. If answer is 4 or -4 code = 3
    5. If answer is $y = 4$ code = 4
    6. If answer is $y = -4$ code = 5

L=Limit C=Continuity D=Differentiation G=Graphing A=Application of Differential Calculus	Co=Conceptual Al=Algorithmic	LD1=Level of Difficulty is Simple LD2=Level of Difficulty is Moderate LD3=Level of Difficulty is Complex

Problem P5
On derivatives
D & L, Al, LD=2

1. Use the Newton's Quotient definition of the derivative to prove that when $f(x)=\frac{2}{x-1}$ then

$$f'(x)=\frac{d\,f(x)}{dx}=\frac{-2}{(x-1)^2}$$

Solution:

$$f(\ \ ) = \frac{1}{2(\ \ )-1}$$

$$f(x+h) = \frac{1}{2(x+h)-1}=\frac{1}{2x+2h-1}$$

$$f(x) = \frac{1}{2(x)-1}=\frac{1}{2x-1}$$

$$f(x+h) - f(x) =$$

$$\frac{1}{2x+2h-1}-\frac{1}{2x-1}=\frac{1}{(2x+2h-1)}\times\frac{(2x-1)}{(2x-1)}-\frac{1}{2x-1}\times\frac{(2x+2h-1)}{(2x+2h-1)}=\frac{(2x-1)-(2x+2h-1)}{(2x+2h-1)(2x-1)}$$

$$= \frac{2x-1-2x-2h+1}{(2x+2h-1)(2x-1)}=\frac{-2h}{(2x+2h-1)(2x-1)}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{\left(\frac{-2h}{(2x+2h-1)(2x-1)}\right)}{h}=\left(\frac{-2h}{(2x+2h-1)(2x-1)}\right)\times\frac{1}{h}=\frac{-2}{(2x+2h-1)(2x-1)}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{-2}{(2x+2h-1)(2x-1)}=\frac{-2}{(2x+2(0)-1)(2x-1)}=\frac{-2}{(2x-1)(2x-1)}=\frac{-2}{(2x-1)^2}$$

Coding: N.B. For coding purposes **code = 99 when leaving blank**

1. Does the student make an attempt to use "Newton's Quotient" definition (even if there are errors):
   1. Yes (at least states the equation for $f\,'(x)$ even if there are errors in the equation) code = 0
   2. Yes, but miscopies the function $f(x)$ code = 1
   3. No, just uses rules of differentiation code = 2
   4. No answer at all code = 3

Fill in the remaining questions **only if** the answer to question 1. above is not code = 3. In all other cases use code = 99 for the remaining questions.

2. Count the number of steps used in the solution: count a step for each new expression on a new line, or after = or after ⇒ , but don't count copying of the original problem as a step if it is the first step.

3. Count of algebra errors:

4. Existence of miracles:
   1. No "miraculous" changes in calculation so that answer comes out correct code = 0
   2. One or more "miraculous" changes in calculation so that answer comes out correct code = 1

Fill in the remaining questions **only if** the answer to question 1. above is not code = 2. In all other cases use code = 99 for the remaining questions.

5. Knowledge of basic Newton's Quotient formula:
   1. whether limit is used or not, or the particular letters exhibited here are used, the student shows knowledge of Newton's Quotient being based on the slope of a secant line, *i.e.*, the ratio $\frac{f(x+h)-f(x)}{h}$, or anything equivalent — code = 0
   2. student believes there is a ratio involved, but has an incorrect formula for the ratio — code = 1
   3. student does not believe that there is a ratio involved — code = 2

Fill in the remaining questions **only if** the answer to question 5. above is code = 0 or code = 1. If the code is 2, then code the remaining questions as code = 99.

6. Does the student know how to compute $f(x+h)$ (or equivalent in other letters):
   1. student correctly computes initial algebraic value of $f(x+h)$, *i.e.*, $\frac{1}{2(x+h)-1}$ or $\frac{1}{2x+2h-1}$ — code = 0
   2. student jumps past this, but work indicates correct understanding of this — code = 1
   3. student jumps past this and there is an error so cannot tell if they understood this — code = 2
   4. student clearly makes an error at this stage, $f(x+h) = f(x) + h$ — code = 3
   5. student does not compute $f(x+h)$ — code = 4

7. Does the student actually use "Newton's Quotient" definition in computing $f'(x)$(even if there are errors):
   1. Yes (no rules of differentiation are used) — code = 0
   2. No, uses rules of differentiation — code = 1

Fill in the remaining questions **only if** the answer to question 7. above is code = 0. If the code is 1, then code the remaining questions as code = 99.

8. Existence of limit in definition of derivative:
   1. student uses limit (even if not formally) in conjunction with Newton's Quotient ratio to compute derivative — code = 0
   2. student does not use limit at all — code = 1

9. If the answer to to question 8. is 1 code the remaining questions 99; Method:
   1. student makes attempt to compute Newton's Quotient ratio, or parts thereof before attempting to apply the limit, but does eventually apply the limit — code = 0
   2. student combines computation of the Newton's Quotient ratio and the limit immediately — code = 1

10. Count the number of times the symbol $\lim_{h \to 0}$ is omitted when it shouldn't be
11. Count the number of times the symbol $\lim_{h \to 0}$ is written when it shouldn't be
12. Count the number of times the symbol $\lim_{h \to 0}$ is written without an expression of any type
13. Count the number of times the symbol lim is written without writing $h \to 0$ underneath.
14. Use of $f(\ )$ notation - writes the first step in Steve's solution:
    1. student uses this notation, at least once — code = 0
    2. student never uses this notation, — code = 1

15. Use of $f(x+h)$ with $x \to 0$ instead of $h \to 0$
    1. Writes and uses $h \to 0$ — code = 0
    2. Writes $x \to 0$ but uses $h \to 0$ — code = 1
    3. Writes and uses $x \to 0$ — code = 2

L=Limit C=Continuity D=Differentiation G=Graphing A=Application of Differential Calculus	Co=Conceptual Al=Algorithmic	LD1=Level of Difficulty is Simple LD2=Level of Difficulty is Moderate LD3=Level of Difficulty is Complex

N.B. Use code = 99 for any questions that you would otherwise leave blank.

Problem P6
On Derivative Rules
D, Al, a)=LD2, b)=LD2, c)=LD2, d)=LD3, e)=LD3

Determine the indicated derivatives in each case:

a) $f(x)=5x^3-\frac{2}{x^2}+x^{2/3}+e^{-2x}$, $f'(x)=\frac{df(x)}{dx}$ and $f''(x)=\frac{d^2f(x)}{dx^2}$

Solution:
We begin by rewriting $f(x)$ slightly so that it will be easier to differentiate. "A spoonful of Algebra/Functions makes the Calculus go down easier."

$$f(x)=5x^3-\frac{2}{x^2}+x^{2/3}+e^{-2x}=5x^3-2x^{-2}+x^{2/3}+e^{-2x}$$

Now we use the Rules of Differentiation and compute the first derivative:

$f'(x)=\frac{d\,f(x)}{dx}=\frac{d\left(5x^3-2x^{-2}+x^{2/3}+e^{-2x}\right)}{dx}=\frac{d\,5x^3}{dx}-\frac{d\,2x^{-2}}{dx}+\frac{d\,x^{2/3}}{dx}+\frac{d\,e^{-2x}}{dx}$	**Sum and Difference Rules**
$=5\frac{d\,x^3}{dx}-2\frac{d\,x^{-2}}{dx}+\frac{2}{3}x^{-1/3}+\frac{d\,e^{-2x}}{d(-2x)}\times\frac{d(-2x)}{dx}$	**Multiplication by a Constant and Chain Rules**
$=5\cdot 3x^2-2(-2)x^{-3}+\frac{2}{3}x^{-1/3}+e^{-2x}\times -2\frac{d\,x}{dx}$	**Power, e$^{(\ )}$ & Multiplication by a Constant Rules**
$=15x^2+4x^{-3}+\frac{2}{3}x^{-1/3}+e^{-2x}\times -2(1)$	**Algebra/Arithmetic and Identity Rule**
$=15x^2+4x^{-3}+\frac{2}{3}x^{-1/3}-2e^{-2x}$	**Algebra/Arithmetic**

Coding for a): (Ignore slight difference in Dawson example)

1. Algebra done prior to differentiation:
   1. Evidence (implicit or explicit) that $-2/x^2$ is done by power rule — code = 0
   2. Evidence (implicit or explicit) that $-2/x^2$ is done by quotient rule — code = 1
   3. No evidence of method used for $-2/x^2$ term — code = 2
   4. No answer provided for this term — code = 3

2. Sum and Difference Rule Skill:
   1. Student (implicitly or explicitly) begins by using sum and difference rules — code = 0
   2. Student attempts to use some other rule of differentiation first — code = 1
   3. Student attempts to use Newton's Quotient — code = 2
   4. Student leaves question blank — code = 3

If answer to 2. is code = 3, remaining questions on a) are code = 99

3. Algebra, Differentiation Algorithm Skill: Count the number of erroneous terms (0 to 4) in answer

4. Chain Rule Skill:
   1. Student correctly uses Chain Rule on term e^{-2x} **(Dawson is sin(-2x) instead)** — code = 0
   2. Student (implicitly or explicitly) attempts to use Chain Rule on term e^{-2x}, but there is at least one error — code = 1
   3. Student does not (implicitly or explicitly) attempt to use Chain Rule on term e^{-2x} — code = 2
5. Power Rule Skill:

1. Student correctly uses Power Rule on term $x^{2/3}$ code = 0
2. Student (implicitly or explicitly) attempts to use Power Rule on term $x^{2/3}$, but there is at least one error code = 1
3. Student does not (implicitly or explicitly) attempt to use Power Rule on term $x^{2/3}$ code = 2

Solution for second derivative:
Now we use the Rules of Differentiation and compute the second derivative:

$f''(x)=\frac{d^2f(x)}{dx^2}=\frac{d\left(\frac{df(x)}{dx}\right)}{dx}=\frac{d\left(15x^2+4x^{-3}+\frac{2}{3}x^{-1/3}-2e^{-2x}\right)}{dx}$	The second derivative is just the first derivative of the first derivative.
$=\frac{d(15x^2)}{dx}+\frac{d(4x^{-3})}{dx}+\frac{d\left(\frac{2}{3}x^{-1/3}\right)}{dx}-\frac{d(2e^{-2x})}{dx}$	Sum and Difference Rules
$=15\frac{dx^2}{dx}+4\frac{dx^{-3}}{dx}+\left(\frac{2}{3}\right)\frac{dx^{-1/3}}{dx}-2\frac{de^{-2x}}{dx}$	Multiplication by a Constant Rule
$=15\cdot 2x+4\cdot(-3)x^{-4}+\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)x^{-4/3}-2\frac{de^{-2x}}{d(-2x)}\times\frac{d(-2x)}{dx}$	Power and Chain Rule
$=30x-12x^{-4}-\left(\frac{2}{9}\right)x^{-4/3}-2e^{-2x}\times(-2)\frac{dx}{dx}$	Algebra/Arithmetic, e( ) and Multiplication by a Constant Rule
$=30x-12x^{-4}-\left(\frac{2}{9}\right)x^{-4/3}-2e^{-2x}\times(-2)(1)$	Algebra/Arithmetic and Identity Rule
$=30x-12x^{-4}-\left(\frac{2}{9}\right)x^{-4/3}+4e^{-2x}$	Algebra/Arithmetic

Coding for a. second derivative:

6. Algebra, Differentiation Algorithm Skill: Count the number of erroneous terms (0 to 4) in answer

7. Understanding of second derivative
   1. Student shows evidence (implicit or explicit) of notion that second derivative is derivative of first derivative code = 0
   2. Student does not show evidence of notion that second derivative is derivative of first derivative code = 1
   3. Student leaves this blank code = 2

N.B. For b) - e) by a trivial error in differentiation we mean a simple recall error such as the derivative of a trig. function, or D(x^3)=2x^2

b) $f(x)=\sin^2(x)\cos(3x),\ f'(x)=\frac{df(x)}{dx}$

Solution:

$f'(x)=\frac{df(x)}{dx}=\frac{d\sin^2(x)\cos(3x)}{dx}=\left(\frac{d\sin^2(x)}{dx}\right)\cos(3x)+\sin^2(x)\left(\frac{d\cos(3x)}{dx}\right)$	**Product Rule**
$=\left(\frac{d(\sin(x))^2}{d(\sin(x))}\times\frac{d(\sin(x))}{dx}\right)\cos(3x)+\sin^2(x)\left(\frac{d\cos(3x)}{d3x}\times\frac{d3x}{dx}\right)$	Chain Rule (twice)
$=(2\sin(x)\times\cos(x))\cos(3x)+\sin^2(x)\left((-\sin(3x)\times 3\frac{dx}{dx}\right)$	Power, **sin( ), cos( )** & Multiplication by a Constant Rules
$=2\sin(x)\cos(x)\cos(3x)-3\sin^2(x)\sin(3x)(1)$	Algebra/Arithmetic & Identity Rule
$=2\sin(x)\cos(x)\cos(3x)-3\sin^2(x)\sin(3x)$	Algebra/Arithmetic

Coding for b):

8. Application of Differentiation Algorithm
   1. Student (implicitly or explicitly) begins by using product rule code = 0
   2. Student attempts to use some other rule of differentiation first code = 1
   3. Student leaves question blank code = 2

If answer to 8. is code = 2, remaining questions on b) are code = 99

9. Product Rule Skill:
   1. Student uses Product Rule correctly (*i.e.*, has pattern or rule correct, even if subsequent differentiation is wrong) code = 0
   2. Student attempts to use Product Rule but has some error in understanding the rule code = 1
   3. Student does not attempt to use the Product Rule code = 2

10. Chain Rule Skill:
    1. Student correctly uses Chain Rule on term $\cos(3x)$ code = 0
    2. Student (implicitly or explicitly) attempts to use Chain Rule on term $\cos(3x)$, but there is at least one error code = 1
    3. Student does not (implicitly or explicitly) attempt to use Chain Rule on term $\cos(3x)$ code = 2

11. Trigonometric Rules Skill:
    1. Student correctly uses Trig. Rules on both $\sin(x)$ and $\cos(3x)$ code = 0
    2. Student makes an error in one Trig. Rule usage, but other is correct code = 1
    3. Student makes errors in both Trig. Rule usages code = 2

12. Differentiation Algorithm Skill:
    1. Student arrives at correct answer code = 0
    2. Student makes one trivial error code = 1
    3. Student makes multiple trivial errors or one substantial error but is essentially on track code = 2
    4. Student is not even close to correct code = 3

c) $f(x)=\dfrac{(x^2-3)^3}{(2x+1)^2},\ f'(x)=\dfrac{df(x)}{dx}$

Solution:

<table>
<tr><td>$$f'(x)=\frac{df(x)}{dx}=\frac{d\left(\frac{(x^2-3)^3}{(2x+1)^2}\right)}{dx}=\frac{\frac{d(x^2-3)^3}{dx}(2x+1)^2-(x^2-3)^3\frac{d(2x+1)^2}{dx}}{\left((2x+1)^2\right)^2}$$</td><td>Quotient Rule</td></tr>
<tr><td>$$=\frac{\left(\frac{d(x^2-3)^3}{d(x^2-3)}\times\frac{d(x^2-3)}{dx}\right)(2x+1)^2-(x^2-3)^3\left(\frac{d(2x+1)^2}{d(2x+1)}\times\frac{d(2x+1)}{dx}\right)}{(2x+1)^4}$$</td><td>Chain Rule</td></tr>
<tr><td>$$=\frac{\left(3(x^2-3)^2\times\left[\frac{dx^2}{dx}-\frac{d3}{dx}\right]\right)(2x+1)^2-(x^2-3)^3\left(2(2x+1)\times\left[\frac{d2x}{dx}+\frac{d1}{dx}\right]\right)}{(2x+1)^4}$$</td><td>Power, Difference and Sum Rules</td></tr>
<tr><td>$$=\frac{\left(3(x^2-3)^2\times[2x-0]\right)(2x+1)^2-(x^2-3)^3\left(2(2x+1)\times\left[2\frac{dx}{dx}+0\right]\right)}{(2x+1)^4}$$</td><td>Power, Constant, Multiplication by a Constant Rules</td></tr>
<tr><td>$$=\frac{\left(6x(x^2-3)^2\right)(2x+1)^2-(x^2-3)^3\left(2(2x+1)\times[2(1)]\right)}{(2x+1)^4}$$</td><td>Algebra/Arithmetic and Identity Rule</td></tr>
<tr><td>$$=\frac{\left(6x(x^2-3)^2\right)(2x+1)^2-(x^2-3)^3\left(4(2x+1)\right)}{(2x+1)^4}$$</td><td>Algebra/Arithmetic</td></tr>
<tr><td>$$=\frac{2(2x+1)(x^2-3)^2\left[3x(2x+1)-2(x^2-3)\right]}{(2x+1)^4}$$</td><td>Algebra/Arithmetic</td></tr>
<tr><td>$$=\frac{2(x^2-3)^2\left[3x(2x+1)-2(x^2-3)\right]}{(2x+1)^3}$$</td><td>Algebra/Arithmetic</td></tr>
</table>

Coding for c):

13. Application of Differentiation Algorithm
    1. Student (implicitly or explicitly) begins by using Quotient Rule — code = 0
    2. Student begins by using algebra to bring denominator into numerator as a negative exponent term, then Product Rule — code = 1
    3. Student attempts to use some rule of differentiation other than described in a. and b. above as first step — code = 2
    4. Student leaves question blank — code = 3

If answer to 13. is code = 3, remaining questions on c) are code = 99

14. Quotient Rule Skill
    1. Student uses Quotient Rule correctly (*i.e.*, has pattern or rule correct, even if subsequent differentiation is wrong) — code = 0
    2. Student attempts to use Quotient Rule but has some error in understanding the rule — code = 1
    3. Student does not attempt to use the Quotient Rule — code = 2

15. Differentiation Algorithm Skill:

    N.B. For a - d, even if the student makes a copy error or algebra error, so long as which rules and the sequence they are to be used in remains unchanged, coding continues as if no error was made.
    1. Student uses the differentiation rules correctly — code = 0
    2. Student makes one error in using the differentiation rules — code = 1
    3. Student makes multiple errors in using the differentiation rules but is essentially on track (sequence and the rules) — code = 2
    4. Student is not even close to correct (uses wrong rules or in wrong sequence) — code = 3
    5. Error in copying or algebra that changes the rules that are used or the sequence in which they are used — code = 4

16. Algebra Skill:
    1. Student uses the algebra rules correctly — code = 0
    2. Student makes only trivial error(s) (*e.g.*, adds up $2x + 4x = 8x$, or forgets a term in subsequent computation, but not such as incorrectly multiplying out a product of bracketed terms) — code = 1
    3. Student uses the algebra rules incorrectly — code = 2

d) $f(x) = (\tan(2x))^{x^2}$, $f'(x) = \dfrac{df(x)}{dx}$

Solution:
We note that this function has a variable base and a variable exponent. Thus, there is no "Rule of Differentiation" that applies. Instead we use the technique known as Logarithmic Differentiation.

$f(x) = (\tan(2x))^{x^2}$	The definition of the function.
$\ln(f(x)) = \ln\left((\tan(2x))^{x^2}\right) = x^2 \ln(\tan(2x))$	Apply ln( ) to both sides so that the exponent inside the ln( ) becomes a factor outside the ln( ).
$\dfrac{d\ln(f(x))}{dx} = \dfrac{d\left(x^2 \ln(\tan(2x))\right)}{dx}$	Differentiate both sides (implicit differentiation now).
$\dfrac{d\ln(f(x))}{df(x)} \times \dfrac{d\,f(x)}{dx} = \left(\dfrac{d\,x^2}{dx}\right)\ln(\tan(2x)) + x^2\left(\dfrac{d\ln(\tan(2x))}{dx}\right)$	Chain and Product Rules
$\dfrac{1}{f(x)} \times \dfrac{d\,f(x)}{dx} = (2x)\ln(\tan(2x)) + x^2\left(\dfrac{d\ln(\tan(2x))}{d(\tan(2x))} \times \dfrac{d\tan(2x)}{d(2x)} \times \dfrac{d\,2x}{dx}\right)$	ln( ), Power and Chain (2 times) Rules
$\dfrac{1}{f(x)} \times \dfrac{d\,f(x)}{dx} = (2x)\ln(\tan(2x)) + x^2\left(\dfrac{1}{\tan(2x)} \times \sec^2(2x) \times 2\dfrac{d\,x}{dx}\right)$	ln( ), tan( ) and Multiplication by a Constant Rules
$\dfrac{1}{f(x)} \times \dfrac{d\,f(x)}{dx} = (2x)\ln(\tan(2x)) + x^2\left(\dfrac{1}{\tan(2x)} \times \sec^2(2x) \times 2(1)\right)$	Identity Rule
$\dfrac{d\,f(x)}{dx} = f(x)\left[2x\ln(\tan(2x)) + \dfrac{2x^2\sec^2(2x)}{\tan(2x)}\right]$	Multiply both sides by $f(x)$ to isolate the derivative on the L.H.S.
$\dfrac{d\,f(x)}{dx} = \left[2x\ln(\tan(2x)) + \dfrac{2x^2\sec^2(2x)}{\tan(2x)}\right](\tan(2x))^{x^2}$	Algebra/Arithmetic and replace $f(x)$ by its original definition

N.B. Dawson example is too different, code = 98 for all of these on Dawson papers

Coding for d):

17. Application of Logarithmic Differentiation Algorithm
    1. Student (implicitly or explicitly) recognizes this as Log. Diff., and carries out **all** Log. Diff. appropriate steps — code = 0
    2. Student (implicitly or explicitly) recognizes this as Log. Diff., and carries out some but not **all** ... steps — code = 1
    3. Student does not recognize this as Log. Diff. and attempts to use some other rule as first step — code = 2
    4. Student leaves question blank — code = 3

If answer to 17. is code = 3, remaining questions on d) are code = 99

18. Trigonometric Rules Skill:
    1. Student correctly uses Trig. Rule on $\tan(x)$ — code = 0
    2. Student makes an error in Trig. Rule usage — code = 1

19. Differentiation Algorithm Skill:
    N.B. For a - d, even if the student makes a copy or algebra error, so long as which rules and the sequence they are to be used in remains unchanged, coding continues as if no error was made.
    1. Student arrives at correct answer code = 0
    2. Student makes one trivial error code = 1
    3. Student makes multiple trivial errors or one substantial error but is essentially on track code = 2
    4. Student is not even close to correct code = 3
    5. Error in copying or algebra that changes the rules that are used or the sequence in which they are used code = 4

20. Algebra Skill:
    1. Student uses the algebra rules correctly code = 0
    2. Student makes only trivial error(s) (*e.g.*, adds up $2x + 4x = 8x$, or forgets a term in subsequent computation, but not such as incorrectly multiplying out a product of bracketed terms) code = 1
    3. Student uses the algebra rules incorrectly code = 2

e) $f(x) = \sqrt{x^3 - \ln(\sec(x))}$, $f'(x) = \dfrac{df(x)}{dx}$

Solution:

$f(x) = \sqrt{x^3 - \ln(\sec(x))} = \left(x^3 - \ln(\sec(x))\right)^{1/2}$	A bit of Algebra to make the Calculus easier.
$f'(x) = \dfrac{df(x)}{dx} = \dfrac{d\left(x^3 - \ln(\sec(x))\right)^{1/2}}{dx} = \dfrac{d\left(x^3 - \ln(\sec(x))\right)^{1/2}}{d\left(x^3 - \ln(\sec(x))\right)} \times \dfrac{d\left(x^3 - \ln(\sec(x))\right)}{dx}$	Chain Rule
$= \left(\dfrac{1}{2}\right)\left(x^3 - \ln(\sec(x))\right)^{-1/2} \times \left[\dfrac{d\,x^3}{dx} - \dfrac{d\ln(\sec(x))}{dx}\right]$	Power and Difference Rules
$= \left(\dfrac{1}{2}\right)\left(x^3 - \ln(\sec(x))\right)^{-1/2} \times \left[3x^2 - \dfrac{d\ln(\sec(x))}{d(\sec(x))} \times \dfrac{d\sec(x)}{dx}\right]$	Power and Chain Rules
$= \left(\dfrac{1}{2}\right)\left(x^3 - \ln(\sec(x))\right)^{-1/2} \times \left[3x^2 - \dfrac{1}{\sec(x)} \times \sec(x)\tan(x)\right]$	ln( ) and sec( ) Rules
$= \dfrac{3x^2 - \tan(x)}{2\sqrt{x^3 - \ln(\sec(x))}}$	Algebra

N.B. Dawson example is too different, code = 98 for all of these on Dawson papers
Coding for e):

21. Algebra done prior to differentiation:
    1. Yes, √ is rewritten as ( )$^{1/2}$ prior to differentiation code = 0
    2. No, but Power Rule is used as if rewritten correctly prior to differentiation code = 1
    3. Something else is done code = 2
    4. Question left blank code = 3

If answer to 21. is code = 3, remaining questions on e) are code = 99

22. Chain Rule Recognition:
    1. Student correctly notes the need to use Chain Rule both for ( )$^{1/2}$ and ln(sec(x)) code = 0
    2. Student correctly notes the need to use Chain Rule for one but not both of ( )$^{1/2}$ and ln(sec(x)) code = 1
    3. Student does not note the need to use Chain Rule in either case code = 2

23. Chain Rule Skill:
    1. Student correctly uses Chain Rule both for ( )$^{1/2}$ and ln(sec(x)) code = 0
    2. Student correctly uses Chain Rule for one but not both of ( )$^{1/2}$ and ln(sec(x)) code = 1
    3. Student does not note use Chain Rule correctly in either case code = 2

24. Differentiation Algorithm Skill:

N.B. For a - d, even if the student makes a copy or algebra error, so long as which rules and the sequence they are to be used in remains unchanged, coding continues as if no error was made.

1. Student arrives at correct answer code = 0
2. Student makes one trivial error code = 1
3. Student makes multiple trivial errors or one substantial error but is essentially on track code = 2
4. Student is not even close to correct code = 3
5. Error in copying or algebra that changes the rules that are used or the sequence in which they are used code = 4

25. Algebra Skill:
    1. Student uses the algebra rules correctly code = 0
    2. Student makes only trivial error(s) (*e.g.*, adds up $2x + 4x = 8x$, or forgets a term in subsequent computation, but not such as incorrectly multiplying out a product of bracketed terms) code = 1
    3. Student uses the algebra rules incorrectly code = 2

L=Limit C=Continuity D=Differentiation G=Graphing A=Application of Differential Calculus	Co=Conceptual Al=Algorithmic	LD1=Level of Difficulty is Simple LD2=Level of Difficulty is Moderate LD3=Level of Difficulty is Complex

N.B. Use code = 99 for any questions that you would otherwise leave blank.

Problem P13
On curve sketching
G & L, Al & Co, LD3

1. Given $f(x) = x^6 - 10x^4$, $f'(x) = 2x^3(3x^2 - 20)$, and $f''(x) = 30x^2(x^2 - 4)$, showing all of your work:
   a) determine any asymptotes and all x and y-intercepts of $f(x)$;
   b) determine on which x-intervals the function $f(x)$ is increasing, on which x-intervals the function $f(x)$ is decreasing, all relative extrema, vertical tangent lines and cusps;
   c) determine on which x-intervals the function $f(x)$ is concave up, on which x-intervals the function $f(x)$ is concave down, and all points of inflection;
   d) sketch a graph of $f(x)$ consistent with the information that you have gathered above.

Solution:
We observe that the given function $f(x)$ is a polynomial of degree larger than 1, hence it will not have any asymptotes, vertical, horizontal or oblique. Nor will it, or its derivative or second derivative have any discontinuities of any type. Thus, there will be no cusps, vertical tangent lines, critical numbers where $f'(x)$ is discontinuous, possible points of inflection where $f''(x)$ is discontinuous.

a) As observed above there are no asymptotes, since $f(x)$ is a polynomial.
   y-intercept: $f(0) = 0$ ---- so this function passes through the origin
   x-intercepts: $f(x) = 0 \Rightarrow x^6 - 10x^4 = 0 \Rightarrow x^4(x^2 - 10) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{10}$
   Edge behaviour: degree 6 polynomial, behaves at the edges like the leading term, in this case x^6, hence at both edges this function heads upwards towards infinity
   alternatively, $\lim_{x\to\pm\infty}\left(x^6 - 10x^4\right) = \lim_{x\to\pm\infty} x^6 = \infty$
   Thus, there are no horizontal asymptotes at the edges.

b) As observed above, since $f'(x)$ is continuous, *i.e.*, it has no discontinuities. Thus, the only critical numbers will be due to the derivative being zero.

   $f'(x) = 0 \Rightarrow 2x^3(3x^2 - 20) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{\frac{20}{3}} = \pm 2\sqrt{\frac{5}{3}}$

   There are four x-intervals created by the three critical numbers. We pick one x-value to represent each interval.

   $f'(-4) = 2(-4)^3(3(-4)^2 - 20) = -128(48 - 20) = -128(28) < 0$, so $f'(x)$ is negative on $\left(-\infty, -2\sqrt{\frac{5}{3}}\right)$, so $f(x)$ is decreasing on that interval

   $f'(-1) = 2(-1)^3(3(-1)^2 - 20) = -2(3 - 20) = -2(-17) = 54 > 0$, so $f'(x)$ is positive on $\left(-2\sqrt{\frac{5}{3}}, 0\right)$, so $f(x)$ is increasing on that interval

   $f'(1) = 2(1)^3(3(1)^2 - 20) = 2(3 - 20) = 2(-17) = -54 < 0$, so $f'(x)$ is negative on $\left(0, 2\sqrt{\frac{5}{3}}\right)$, so $f(x)$ is decreasing on that interval

   $f'(4) = 2(4)^3(3(4)^2 - 20) = 128(48 - 20) = 128(28) > 0$, so $f'(x)$ is positive on $\left(2\sqrt{\frac{5}{3}}, \infty\right)$, so $f(x)$ is increasing on that interval

Based on the above information we note that $f(x)$ must have a local minimum at $x = -2\sqrt{\frac{5}{3}}$, a local maximum at $x = 0$, and a local minimum at $x = 2\sqrt{\frac{5}{3}}$.

c) Since $f''(x)$ is a polynomial it is continuous everywhere, *i.e.*, it has no discontinuities. Thus, the only possible points of inflection will be due to the second derivative being zero.

$f''(x) = 0 \Rightarrow 30x^2(x^2 - 4) = 0 \Rightarrow x = -2, 0, 2$

There are four x-intervals created by the three possible points of inflection. We pick one x-value to represent each interval.

$f''(-3) = 30(-3)^2((-3)^2 - 4) = 30(9)(5) > 0$, so $f''(x)$ is positive on $(-\infty,-2)$, so $f'(x)$ is increasing on $(-\infty,-2)$, and $f(x)$ is concave up on $(-\infty,-2)$

$f''(-1) = 30(-1)^2((-1)^2 - 4) = 30(1)(-3) < 0$, so $f''(x)$ is negative on $(-2,0)$, so $f'(x)$ is decreasing on $(-2,0)$, and $f(x)$ is concave down on $(-2,0)$

$f''(1) = 30(1)^2((-1)^2 - 4) = 30(1)(-3) < 0$, so $f''(x)$ is negative on $(0,2)$, so $f'(x)$ is decreasing on $(0,2)$, and $f(x)$ is concave down on $(0,2)$

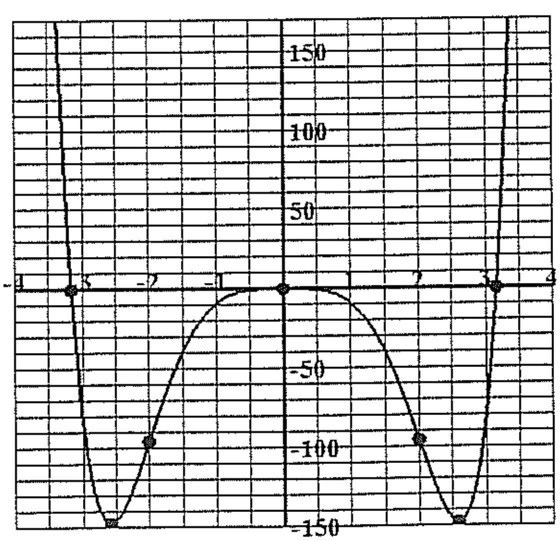
$f''(3) = 30(3)^2((3)^2 - 4) = 30(9)(5) > 0$, so $f''(x)$ is positive on $(2,\infty)$, so $f'(x)$ is increasing on $(2,\infty)$, and $f(x)$ is concave up on $(2,\infty)$

Based on the information above, $f(x)$ has points of inflection at $x = -2$ and $x = 2$.
We have all the information that we computed above recorded in a table.

	no H A		x-int		m		PI		M x-int y-int		PI		m		x-int		no H A
x	$-\infty$		$-\sqrt{10}$		$-2\sqrt{\frac{5}{3}}$		-2		0		2		$2\sqrt{\frac{5}{3}}$		$\sqrt{10}$		∞
$f(x)$	∞	↘	0 ↘	↘	-148 –	↗	-96 ↗	↗	– 0	↘	-96 ↘	↘	-148 –	↗	0 ↗	↗	∞
$f'(x)$		-	-	-	0	+	+	+	0	-	-	-	0	+	+	+	
$f''(x)$		+	+	+	+	+	0	-	0	-	0	+	+	+	+	+	
		⌣	⌣	⌣	⌣	⌣		⌢		⌢		⌣	⌣	⌣	⌣	⌣	

As we record the information previously gathered, we note the points at which the function has a local minimum, local maximum and points of inflection. At this point we compute the approximate values of f at the minimum, maximum and points of inflection and add this information to the table.

Looking at the entire table we note that the information gathered is consistent, so we sketch a graph.



Coding for P13

Algorithm:

1. Do they check for edge behaviour of f?
   1. Yes - using limits
   2. Yes - by some other method
   3. Unclear
   4. Missing *i.e.*, No evidence that they checked for edge behaviour.

If they answered 3. or 4. enter 99 for 2.

2. When they checked for edge behaviour of f, did they get it correct?
   1. Yes
   2. No

3. Do they follow sequence: determine zeroes of f', followed by computing sign of f' on all intervals?
   1. Yes
   2. No
   3. Unclear
   4. Missing

4. Do they follow sequence: determine zeroes of f'', followed by computing sign of f'' on all intervals?
   1. Yes
   2. No
   3. Unclear
   4. Missing

Arithmetic/Algebra:

5. Count of arithmetic/algebra errors:
6. Count of errors in notation

Graph:

Graph: Existence

7. Presence/absence of graph
   1. Complete Graph Present
   2. Incomplete Graph Present
   99. Absent

If 7. above is coded as 99, then 8. - 14. coded as 99.

If 1. above is coded as 3 or 4, then code 8. below as 99.

Graph: Edge behaviour

8. Does the edge behaviour on the graph match what student determined by other method(s)?
   1. Yes
   2. No
   3. Unclear
   99. Edge behaviour is missing or has nothing prior to match up to

N.B. In cases where f' has closely adjacent zero and extremum, if student clearly (*e.g.*, draws a dotted vertical line) distinguishes between these two, then we give more generous latitude to error of horizontal placement of one of the two corresponding features of f.

Graph: Sign of f' corresponds to direction of f

9. For each zero of f', is a corresponding (with reasonable margin of error) Extrema/Stationary Point of f drawn on the graph: Count # of omissions (if only one zero of f', code 99)

10. For each zero of f', is the correct corresponding (with reasonable margin of error) Extrema/Stationary Point of f drawn on the graph: Count # of errors or omissions (if only one zero of f', code 99)

11. Alternating pattern on graph of intervals of increase and decrease of f:
    0. Correct
    1. Error(s)

Graph: Sign of f'' corresponds to concavity of f

12. For each zero of f'', is a corresponding (with reasonable margin of error) Point of Inflection of f drawn on the graph: Count # of omissions (if only one zero of f'', code 99)

13. For each zero of f'', is the correct corresponding (with reasonable margin of error) Point of Inflection of f drawn on the graph: Count # of errors or omissions (if only one zero of f'', code 99)

14. Alternating pattern of intervals of concavity on graph of f:
    0. Correct
    1. Error(s)

Explanations

Explanations: Edge Behaviour

15. Student makes statement(s) about edge behaviour of the function f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

Explanations: Continuity/Discontinuity

16. Student makes statement(s) about continuity/discontinuity of the function f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

Explanations: Connection between sign of f' and direction of f

17. Verbal Explanation of relationship between sign (-/+) of f' and direction (↘/↗) of f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

18. Symbolic Explanation of relationship between sign (-/+) of f' and direction (↘/↗) of f. In different classes this takes on different appearances. It may be tabular, with a line for f and a line for f' (and a correspondence between direction of f and sign of f' established by +/↗ and -/↘) or it may be an x-axis, with arrows and signs or other symbols, or it may be in symbolic statements including intervals. The key is the establishment of the correspondence between the direction of f and the sign of f' using symbols, possibly symbols and a few words.
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

19. Verbal Explanation of relationship between zeroes (0) of f' and extrema or stationary points of f
    1. only correct statements
    2. correct but omit mention of stationary point possibility
    3. mixture of correct and incorrect statements
    4. only false statement(s)
    99. no statement

20. Symbolic Explanation of relationship between zeroes (0) of f' and extrema or stationary points of f. In different classes this takes on different appearances. It may be tabular, with a line for f and a line for f' (and a correspondence between a 0 of f' and a M, m, SP), or it may be an x-axis, with arrows and signs or other symbols.
    1. only correct statement(s)
    2. correct but omit mention of stationary point possibility
    3. mixture of correct and incorrect statements
    4. only false statement(s)
    99. no statement

Explanations: Connection between sign of f'' and concavity of f

21. Verbal Explanation of relationship between sign (-/+) of f'' and concavity (⌒/⌣) of f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

22. Symbolic Explanation of relationship between sign (-/+) of f'' and concavity (⌒/⌣) of f. In different classes this takes on different appearances. It may be tabular, with a line for f and a line for f'' (and a correspondence between concavity of f and sign of f'' established by +/⌣ and -/⌒) or it may be an x-axis, with symbols and signs, or it may be in symbolic statements including intervals. The key is the establishment of the correspondence between the direction of f and the sign of f'' using symbols, possibly symbols and a few words.
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

23. Verbal Explanation of relationship between zeroes (0) of f'' and points of inflection of f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

24. Symbolic Explanation of relationship between zeroes (0) of f'' and points of inflection of f. In different classes this takes on different appearances. It may be tabular, with a line for f and a line for f'' (and a correspondence between points of inflection of f and change of sign of f'' or it may be an x-axis, with symbols and signs, or it may be in symbolic statements including intervals. The key is the establishment of the correspondence between the direction of f and the sign of f'' using symbols, possibly symbols and a few words.
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

L=Limit C=Continuity D=Differentiation G=Graphing A=Application of Differential Calculus	Co=Conceptual Al=Algorithmic	LD1=Level of Difficulty is Simple LD2=Level of Difficulty is Moderate LD3=Level of Difficulty is Complex

N.B. Use code = 99 for any questions that you would otherwise leave blank.

Problem P14
On curve sketching
G & L, Al & Co, LD3

Given $f(x)=\frac{x+2}{x^3}$, $f'(x)=\frac{-2x-6}{x^4}$ and $f''(x)=\frac{6x+24}{x^5}$, showing all of your work:

a) determine any asymptotes and all x and y-intercepts of $f(x)$;
b) determine on which x-intervals the function $f(x)$ is increasing, on which x-intervals the function $f(x)$ is decreasing, all relative extrema, vertical tangent lines and cusps;
c) determine on which x-intervals the function $f(x)$ is concave up, on which x-intervals the function $f(x)$ is concave down, and all points of inflection;
d) sketch a graph of $f(x)$ consistent with the information that you have gathered above.

Solution:
We observe that the given function $f(x)$ is a rational function, hence it may have asymptotes, vertical, horizontal or oblique. Also, its derivative or second derivative are also rational functions and so may have discontinuities.

a) y-intercept: $f(0)=\text{"}\frac{(0)+2}{(0)^3}\text{"}=\text{"}\frac{2}{0}\text{"}=\text{U}$ - the function has no y-intercept, instead it has an infinite discontinuity (the graph has a vertical asymptote at $x=0$)

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{x+2}{x^3}=\text{"}\frac{2}{0^-}\text{"}=-\infty$, so on the left of $x=0$ the graph will head downward towards $-\infty$

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{x+2}{x^3}=\text{"}\frac{2}{0^+}\text{"}=\infty$, so on the right of $x=0$ the graph will head upward towards ∞

x-intercept: $f(x)=0 \Leftrightarrow x+2=0 \Leftrightarrow x=-2$

N.B. We note that there are no other values of x that make the denominator zero, other than $x=0$, hence $f(x)$ has no other discontinuities.

Edges: $\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x+2}{x^3}=\lim_{x\to\pm\infty}\frac{x}{x^3}=\lim_{x\to\pm\infty}\frac{1}{x^2}=\text{"}\frac{1}{\infty}\text{"}=0^+$, so at both edges the function is asymptotic (horizontal asymptote) to the horizontal line $y=0$, *i.e.*, the x-axis. Since we observe that at both edges the function will have positive values, any graph we sketch must approach the x-axis from above at both edges.

b) Critical Numbers:

$f'(x)$ is discontinuous: this only occurs if the denominator, x^4, is zero, *i.e.*, at $x=0$. We already knew this since the function itself is discontinuous at this value of x, hence the derivative cannot exist there.

$f'(x)=0 \Leftrightarrow -2x-6=0 \Leftrightarrow x=-3$

Since there are two critical numbers, we have three intervals to investigate: $(-\infty,-3)$, $(-3,0)$, and $(0,\infty)$.

$f'(-4)=\frac{-(2(-4)+6)}{(-4)^4}=\frac{2}{256}>0$, thus $f'(x)$ is positive on $(-\infty,-3)$, and $f(x)$ is increasing on $(-\infty,-3)$

$f'(-1)=\frac{-(2(-1)+6)}{(-1)^4}=\frac{-4}{1}<0$, thus $f'(x)$ is negative on $(-3,0)$, and $f(x)$ is decreasing on $(-3,0)$

$f'(1)=\frac{-(2(1)+6)}{(1)^4}=\frac{-8}{1}<0$, thus $f'(x)$ is negative on $(0,\infty)$, and $f(x)$ is decreasing on $(0,\infty)$

Based on the information above we note that $f(x)$ has a local maximum at $x=-3$, but $x=0$ is a vertical asymptote as seen in (a) above, so there is no local extremum there.

c) Possible Points of Inflection:

$f''(x)$ is discontinuous: this only occurs if the denominator, x^5, is zero, *i.e.*, at $x = 0$. We already knew this since the function itself is discontinuous at this value of x, hence the derivative cannot exist there, and so the second derivative cannot exist there either.

$f''(x) = 0 \Leftrightarrow 6x + 24 = 0 \Leftrightarrow x = -4$

Since there are two possible points of inflection, we have three intervals to investigate: (-∞,-4), (-4,0), and (0,∞).

$f''(-5) = \dfrac{6(-5)+24}{(-5)^5} = \dfrac{-6}{-3125} = \dfrac{6}{3125} > 0$, thus $f''(x)$ is positive on (-∞,-4), hence $f'(x)$ is increasing on (-∞,-4), and $f(x)$ is concave up on (-∞,-4)

$f''(-1) = \dfrac{6(-1)+24}{(-1)^5} = \dfrac{18}{-1} = -18 < 0$, thus $f''(x)$ is negative on (-4,0), hence $f'(x)$ is decreasing on (-4,0), and $f(x)$ is concave down on (-4,0)

$f''(1) = \dfrac{6(1)+24}{(1)^5} = \dfrac{30}{1} = 30 > 0$, thus $f''(x)$ is positive on (0,∞), hence $f'(x)$ is increasing on (0,∞), and $f(x)$ is concave up on (0,∞)

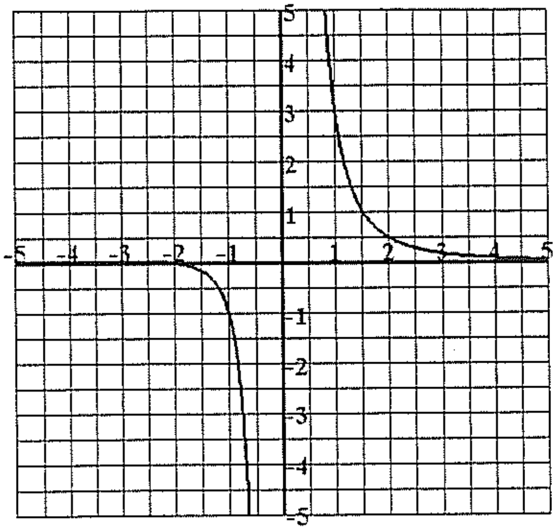
Based on the information above we note that $f(x)$ has a point of inflection at $x = -4$, but $x = 0$ is a vertical asymptote as seen in (a) above, so there is no point of inflection there.

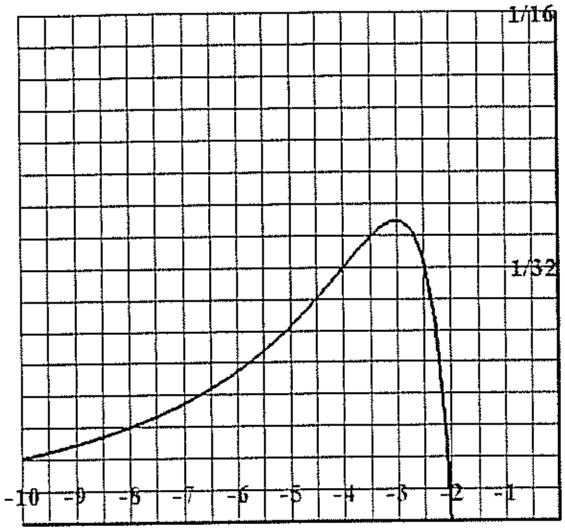
We have all the information that we computed above recorded in a table.

	HA		PI		*M*		*x*-int			VA			HA
x	-∞		-4		-3		-2		0^-	0	0^+		∞
$f(x)$	0^+	↗	↗ 1/32	↗	– 1/27	↘	↘ 0	↘	↘ -∞	U	∞ ↖	↘	0^+
$f'(x)$		+	+	+	0	-	-	-		U		-	
$f''(x)$		+	0	-	-	-	-	-		U		+	
		⌣		⌢	⌢	⌢	⌢	⌢				⌣	

As we record the information previously gathered, we note the points at which the function has a local minimum, local maximum and points of inflection. At this point we compute the approximate values of f at the minimum, maximum and points of inflection and add this information to the table.

Looking at the entire table we note that the information gathered is consistent, so we sketch a graph. Since the values of $f(x)$ at the point of inflection and the local maximum are very small, and there is a vertical asymptote at $x = 0$, this is a difficult graph to portray in a single view. That is, if we make the range of y-values shown large, we will obtain a clear image of the function as x approaches 0, *i.e.*, nearby its vertical asymptote, but we will not see the point of inflection and the local maximum. On the other hand, if we make the range of y-values small, we will obtain a nice image of the point of inflection and the local maximum, but will not see the vertical asymptote. Thus, I have created two graphs. One shows the overall shape, including the vertical asymptote and horizontal asymptotes. The second, zooming in near the point of inflection and local maximum, shows these two aspects particularly well.





Coding for P14

Algorithm:

1. Do they check for edge behaviour of f?
   1. Yes - using limits
   2. Yes - by some other method
   3. Unclear
   4. Missing, *i.e.*, No evidence that they checked for edge behaviour.

If they answered 3. or 4. enter 99 for 2.

2. When they checked for edge behaviour of f, did they get it correct?
   1. Yes
   2. No

3. Do they check for V.A. behaviour of f?
   1. Yes - using limits
   2. Yes - by some other method
   3. Unclear
   99. Missing, *i.e.*, No evidence that they checked for VA behaviour.

If they answered 3. or 4. enter 99 for 4.

4. When they checked for VA behaviour of f, did they get it correct?
   1. Yes
   2. No

5. Do they compute y_{max} value?
   1. Correct
   2. Incorrect
   99. Missing computation

6. Do they follow sequence: determine zeroes of f', followed by computing sign of f' on all intervals?
   1. Yes
   2. No
   3. Unclear
   99. Missing

7. Do they follow sequence: determine zeroes of f'', followed by computing sign of f'' on all intervals?
   1. Yes
   2. No
   3. Unclear
   99. Missing

<u>Symbolic representation:</u>

8. Count of arithmetic/algebra errors:
9. Count of errors in notation

<u>Graph:</u>

Graph: Existence

10. Presence/absence of graph
    1. Complete Graph Present
    2. Incomplete Graph Present
    99. Absent

<u>If 10. above is coded as 99, then code 11. - 17. coded as 99.</u>

<u>If 1. above is coded as 3 or 4, then code 11. below as 99.</u>

Graph: Edge behaviour

11. Does the edge behaviour on the graph match what student determined by other method(s)?
    1. Yes
    2. No
    3. Unclear
    99. Edge behaviour is missing or has nothing prior to match up to

<u>If 3. above is coded as 3 or 4, then code 12. below as 99.</u>

Graph: VA behaviour

12. Does the VA behaviour on the graph match what student determined by other method(s)?
    1. Yes
    2. No
    3. Unclear
    99. VA behaviour is missing or has nothing prior to match up to

<u>N.B.</u> In cases where f' has closely adjacent zero and extremum, if student clearly (*e.g.*, draws a dotted vertical line) distinguishes between these two, then we give more generous latitude to error of horizontal placement of one of the two corresponding features of f.

Graph: Sign of f' corresponds to direction of f

13. For each zero of f', is a corresponding (with reasonable margin of error) Extrema/Stationary Point of f drawn on the graph: Count # of omissions

14. For each zero of f', is the correct corresponding (with reasonable margin of error) Extrema/Stationary Point of f drawn on the graph: Count # of errors or omissions

15. Alternating pattern on graph of intervals of increase and decrease of f:
    0. Correct
    1. Error(s)

Graph: Direction of f' corresponds to concavity of f

16. For each change in direction of f', is a corresponding (with reasonable margin of error) Point of Inflection of f drawn: Count # of omissions

17. For each change in direction of f', is the correct corresponding (with reasonable margin of error) Point of inflection of f drawn on the graph: Count # of errors or omissions

18. Alternating pattern of intervals of concavity on graph of f:
    0. Correct
    1. Error(s)

Vertical scale of the graph

19. Is the vertical position of the maximum (with reasonable margin of error) drawn on the graph
    0. Correct
    1. Incorrect

20. Is the vertical position of the HA (with reasonable margin of error) drawn on the graph
    0. Correct
    1. Incorrect

<u>Explanations</u>

Explanations: Edge Behaviour (HA)

21. Student makes statement(s) about edge behaviour (HA) of the function f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

Explanations: Continuity

22. Student makes statement(s) about continuity of the function f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

Explanations: Discontinuity (VA)

23. Student makes statement(s) about discontinuity (VA) of the function f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

Explanations: Connection between sign of f' and direction of f

24. Verbal Explanation of relationship between sign (-/+) of f' and direction (↘/↗) of f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

25. Symbolic Explanation of relationship between sign (-/+) of f' and direction (↘/↗) of f. In different classes this takes on different appearances. It may be tabular, with a line for f and a line for f' (and a correspondence between direction of f and sign of f' established by +/↗ and -/↘) or it may be an x-axis, with arrows and signs or other symbols, or it may be in symbolic statements including intervals. The key is the establishment of the correspondence between the direction of f and the sign of f' using symbols, possibly symbols and a few words.
  1. only correct statement(s)
  2. mixture of correct and incorrect statements
  3. only false statement(s)
  99. no statement

26. Verbal Explanation of relationship between zeroes (0) of f' and extrema or stationary points of f
  1. only correct statements
  2. correct but omit mention of stationary point possibility
  3. mixture of correct and incorrect statements
  4. only false statement(s)
  99. no statement

27. Symbolic Explanation of relationship between zeroes (0) of f' and extrema or stationary points of f. In different classes this takes on different appearances. It may be tabular, with a line for f and a line for f' (and a correspondence between a 0 of f' and a M, m, SP), or it may be an x-axis, with arrows and signs or other symbols.
  1. only correct statement(s)
  2. correct but omit mention of stationary point possibility
  3. mixture of correct and incorrect statements
  4. only false statement(s)
  99. no statement

Explanations: Connection between sign of f'' and concavity of f

28. Verbal Explanation of relationship between sign (-/+) of f'' and concavity (⌒/⌣) of f
  1. only correct statement(s)
  2. mixture of correct and incorrect statements
  3. only false statement(s)
  99. no statement

29. Symbolic Explanation of relationship between sign (-/+) of f'' and concavity (⌒/⌣) of f. In different classes this takes on different appearances. It may be tabular, with a line for f and a line for f'' (and a correspondence between concavity of f and sign of f'' established by +/⌣ and -/⌒) or it may be an x-axis, with symbols and signs, or it may be in symbolic statements including intervals. The key is the establishment of the correspondence between the direction of f and the sign of f'' using symbols, possibly symbols and a few words.
  1. only correct statement(s)
  2. mixture of correct and incorrect statements
  3. only false statement(s)
  99. no statement

30. Verbal Explanation of relationship between zeroes (0) of f'' and points of inflection of f
  1. only correct statement(s)
  2. mixture of correct and incorrect statements
  3. only false statement(s)
  99. no statement

31. Symbolic Explanation of relationship between zeroes (0) of f'' and points of inflection of f. In different classes this takes on different appearances. It may be tabular, with a line for f and a line for f'' (and a correspondence between points of inflection of f and change of sign of f'' or it may be an x-axis, with symbols and signs, or it may be in symbolic statements including intervals. The key is the establishment of the correspondence between the direction of f and the sign of f'' using symbols, possibly symbols and a few words.

1. only correct statement(s)
2. mixture of correct and incorrect statements
3. only false statement(s)
99. no statement

32. Symbolic Explanation of relationship between zeroes (0) of f'' and points of inflection of f. In different classes this takes on different appearances. It may be tabular, with a line for f and a line for f'' (and a correspondence between points of inflection of f and change of sign of f'' or it may be an x-axis, with symbols and signs, or it may be in symbolic statements including intervals. The key is the establishment of the correspondence between the direction of f and the sign of f'' using symbols, possibly symbols and a few words.

1. only correct statement(s)
2. mixture of correct and incorrect statements
3. only false statement(s)
99. no statement

Applicable across the entire scoring schema for PAREA coding. If we encounter a blank("") or "xx" or "x" or 99, we change the data to blanks.

P3 a)

Characteristic	Coding Schema #'s	Logic of scoring
	#1.	If #1.=1 or #1.=2, then all others are scored as blank.
symbolic skill	#2., 3., 4., 5. & 6.	Make frequency table of values of sum((#3.-#6.)/#2.) and then use it to create scoring table. Remark from the frequency table (frequence table value= "v"): If v=0, then score as 1, else if v<=1/4, then score as 2, else if v<=1/2, then score as 3, else if v<=3/4, then score as 4, else if v>3/4, then score as 5.
knowledge of algorithm	#7.	If #7.=0, then score as 1, else score as 5.
correct answer	#8.	If (#1=1 or #1=2), then 5, else if #8.=0, then score as 1, else if #8.=1, then score as 2, else if #8.=2, then score as 3, else if #8.=3, then score as 5.
conceptual	#9.	If #9.=0, then score as 1, else if #9.=1, then score as 5.
discussion	#10.	If #10.=0, then score as 1, else if #10.=1, then score as 3, else if #10.=2, then score as 5, else if #10.=3, then score as 99.
correct solution	based on score in symbolic (S), algorithmic (A), correct answer (C)	If sum(S+A+C) < 4, then score 1, else if 3 < sum(S+A+C) < 6, then score 2, else if 5 < sum(S+A+C) < 8, then score 3, else if 7 < sum(S+A+C) < 15, then score 4, else if sum(S+A+C) = 15, then score 5.

P3 b)

Characteristic	Coding Schema #'s	Logic of scoring
	#1.	If #1.=1, then all others are scored as 99.
symbolic skill S_1	#2., 3., 4., 5. & 6.	Make frequency table of values of sum((#3.-#6.)/#2.) and then use it to create scoring table
knowledge of algorithm	#7.	If (#11="" or #17= "" or #11>0), then "" if #7.=0, then score as 1, else score as 5.
correct answer	#8.	If #8.=0, then score as 1, else if #8.=1, then score as 2, else if #8.=2, then score as 3, else if #8.=3, then score as 5.
symbolic skill S_2	#9.	If #9.=0, then score as 1, else if #9.=1, then score as 5.
discussion	#10.	If #10.=0, then score as 1, else if #10.=1, then score as 3, else if #10.=2, then score as 5, else if #10.=3, then score as 99.
total symbolic skill	#2. - #6., #9	If S_2=1 then score as S_1, else score as 5
correct solution	based on score in symbolic (S), algorithmic (A) and correct answer (C)	If sum(S+A+C) < 4, then score 1, else if 3 < sum(S+A+C) < 6, then score 2, else if 5 < sum(S+A+C) < 8, then score 3, else if 7 < sum(S+A+C) < 15, then score 4, else if sum(S+A+C) = 15, then score 5.

P3 combined

Characteristic	Coding Schema #'s	Logic of scoring
symbolic skill	all 2 symbolic skill values from P3a and P3b	sum of 2 values/2
knowledge of algorithm	2 knowledge of algorithm values from P3a and P3b	sum of 2 values/2
correct answer	2 correct answer values	sum of 2 values/2
discussion	2 discussion values	sum of 2 values/2
correct solution	2 correct solution values	sum of 2 values/2
concept	#9.	Score the same as conceptual in P3a)

P4a)

Characteristic	Coding Schema #'s	Logic of scoring
correct answer	#1. - #2.	If #1>1 then 5 else if #1=0 then 1 else if (#1=1 and #2=0) then 1 else 3
discussion	#3	If #3=3, then 99 else 2 *#3+1
correct solution	#1 - #2	same as correct answer

P4b)

Characteristic	Coding Schema #'s	Logic of scoring
correct answer	#4.	If #4.< 2 then #4+1 else 5
discussion	#5.	If #5.=3 then 99 else 2*#5.+1
correct solution	#4.	score the same as correct answer

P4c)

Characteristic	Coding Schema #'s	Logic of scoring
correct answer	#6., 7.	If #6.>1 then 5 else if #6.=0 then 1 else if #6.=1 and #7.=0 then 1 else 3
discussion	#8.	If #8.=3 then 99 else 2*#8.+1
correct solution	#6., 7.	score the same as correct answer

P4d)

Characteristic	Coding Schema #'s	Logic of scoring
correct answer	#9. - 11.	If #9.>2 then 5 else if #9.=0 then 1 else if #9.=1 and #10.=0 then 1 else if #9.=2 and #11.=1 then 2 else 3
discussion	#12.	If #12.=3 then 99 else 2*#12.+1
correct solution	#9. - 11.	score the same as correct answer

P4e)

Characteristic	Coding Schema #'s	Logic of scoring
correct answer	#14., 16., 18.	If #14.<2 then c1=1 else if #14.=2 then c1=2 else if #14.=3 then c1=3 else c1=5 If #16.<2 then c2=1 else if #16.=2 then c2=2 else if #16.=3 then c2=3 else c2=5 If #18.<2 then c3=1 else if #18.=2 then c3=2 else if #18.=3 then c3=3 else c3=5 c=(c1+c2+c3)/3
symbolic	#13. - 18.	If #13.=1 then s1=5 else if #14.=0 then s1=1 else if #14.=1 then s1=2 else s1=#14. If #15.=1 then s2=5 else if #16.=0 then s2=1 else if #16.=1 then s2=2 else s2=#14. If #17.=1 then s3=5 else if #18.=0 then s3=1 else if #18.=1 then s3=2 else s3=#14. s=(s1+s2+s3)/3
correct solution	#9. - 11.	score the same as correct answer
discussion	#3,#5,#8,#12	sum of scores from P4a)-P4d)/4

P4f) Combined

Characteristic	Coding Schema #'s	Logic of scoring
Correct Answer	all 5 correct answer values from P4a -P4e	sum of 5 values/5
Discussion	all 4 discussion values from P4a-P4d	sum of 4 values/4
Correct solution	all 5 correct solution values from P4a -P4e	sum of 5 values/5
Symbolic		score the same as symbolic from P4e)

P5

Characteristic	Coding Schema #'s	Logic of scoring
algebra	#2., 3.	Score is #3./#2.
symbolic	#2., 10. - 13., 15.	Score is (#10.+#11.+#12.+#13.+#15.)/#2

correct solution	#2. - 8., 10. - 13., 15.	Let a=algebra score, s=symbolic score
		If (#4.+#5.+#7.+#8.+#15.+(#6.*(#6.-1)))=0 then
		if a+s=0, then 1 else
		if (s=0 and a>0) then 2 else
		if (a=0 and s>0) then 3 else
		if (a>0 and s>0) then 4
		else if (#4.+(#5.-1)+#7.+#8.+#15.+(#6.*(#6.-1)))=0 then
		if a+s=0, then 2 else
		if (s=0 and a>0) then 3 else
		if (a=0 and s>0) then 4 else 5

conceptual	#1.	If (#1.-1)*(#1.)=0 then 1 else 5

P6 a)

Characteristic	Coding Schema #'s	Logic of scoring
correct answer	#3., #6.	Score is (sum(#3.+#6)/2)+1
algorithmic	#2., 4., 5.	If #2.=0, then if #4.+#5.=0 then score as 1, else if #4.+#5.=1, then score as 2, else if #4.+#5.=2, then score as 3, else if #4.+#5.=3, then score as 4, else score as 5.
conceptual	#7.	If #7.=0, then score as 1, else score as 5.
correct solution	#2., 4., 5.	use score from algorithmic

P6 b)

Characteristic	Coding Schema #'s	Logic of scoring
correct answer	#12.	If #12.=3, then score is 5, else score is #12.+1
algorithmic	#8. - 11.	Let s=#9.+#10.+#11. If #8.=0, then if s=0 then score as 1, else if #9.+#10.=0 and #11.>0, then score as 2, else if #9.+#10.=1, then score as 3, else if #9.=#10.=1, then score as 4, else score as 5.
correct solution	#8. - 11.	Use algorithmic score

P6 c)

Characteristic	Coding Schema #'s	Logic of scoring
correct answer	#15., 16.	If #15.=0, then if #16.=0, then score is 1, else if #16.=1, then score is 2, else if #15.=1, then if #16.<2, then score is 3, else if #16.=2 then score is 4, else score is 5.
algorithmic	#13. - 14.	If #13=0, then If #14.=0, then score as 1, else if #14.=1, then score as 2, else else if #13=1, then if #15.=0, then score as 1 else score is 5.
correct solution	#8. - 11.	Use algorithmic score

P6 d)

Characteristic	Coding Schema #'s	Logic of scoring
correct answer	#19.	If #19.>2 then score 5, else score=#19.+1.
algorithmic	#17. & 18.	If #17= blank then blank, else If #17.>1 then 5, else If #18=97, then (#17+1) else score=#17.+#18.+1
algebraic	#20.	Score=2*#20.+1.
correct solution	#17. - 19.	If #17.>1 then 5, else if #19.>2 then 5, else if #17+#18.+#19.=0 then 1, else if #17.=1 and #18.+#19=0 then 2, else score #17.+#18.+#19.

P6 e)

Characteristic	Coding Schema #'s	Logic of scoring
correct answer	#24.	If #24= blank then blank else If #24.>2 then 5 else score=#24.+1.
algorithmic	#21. - 23.	If (#21=blank or #22=blank or #24=blank) then blank else if #24.=4 then blank else if #21.>1 then 5 else if #22.=2 then 5 else if #22.=0 then score=#23.+1 else if #22.=1 then score=#23.+2
algebraic	#25.	If #24.=4 then 99 else score=2*#25.+1.

correct solution	#21. - 24.	If #24.=2 then
		if (#21.>1 and #22.+#23.=0) then 3 else
		score=4
		else score=correct answer score

P6 summary: Scoring Differentiation

Characteristic	Coding Schema #'s	Logic of scoring
algorithmic	#21. - 23.	algorithm=a(section#) If (aP6a=blank or aP6b=blank or aP6c=blank or aP6d=blank or aP6e=blank) then blank else score=Average of (aP6a, aP6b, aP6c, aP6d, aP6e)
correct solution	#21. - 24.	score same as algorithmic
correct answer	all 5 correct answer values from P6a-P6e	sum of 5 values/5
conceptual	P6a	score the same as conceptual P6a
algebraic	all 2 algebraic values from P6d-P6e	sum of 2 values/2

P13

Characteristic	Coding Schema #'s	Logic of scoring
Algorithm	#1.-4.	If #2.=1 then e=1 else e=5 If #3.=1 then fd=1 else fd=5 If #4.=1 then sd=1 else sd=5 Let a=(e+fd+sd)/3
Arithmetic/ Algebra	#5.	score=code
Symbolic	#6.	score=code
Graph	#2., #7., #10.- #11, #13 - #14.	If #2.*#7.=1 & #10.+#11.+#13.+#14.=0 then 1 else 5
Explanation Frequency (ef1)	#15.-24.	If all of #15. - #24 = 99, then 5, else 1
Number of Explanations (ef2)	#15. - #24.	If ef1=5 then 99 else (if #15.=1 then 1 else 0)+(if #16.=1 then 1 else 0)+(if #17. or #18.=1 then 1 else 0)+(if #19. or #20.=1 then 1 else 0)+(if #21. or #22.=1 then 1 else 0)+(if #23. or #24.=1 then 1 else 0)

P14

Characteristic	Coding Schema #'s	Logic of scoring
Algorithm	#1.-7.	If #2.=1 then e=1 else e=5 If #3.<3 then va1=1 else va1=5 If #4.=1 then va2=1 else va2=5 Let va=(va1+va2)/2 If #5.=1 then m=1 else if #5.=2 then m=3 else m=5 If #6.=1 then fd=1 else fd=5 If #7.=1 then sd=1 else sd=5 Let a=(e+va+m+fd+sd)/5
Arithmetic/ Algebra	#8.	score=code
Symbolic	#9.	score=code
Graph	#10.-20.	Let s=#14.+#15.+#17.+#18.+#19.+#20. Let t=#15.+#18.+#19.+#20.+(if #14.=0 then 0 else 1)+(if #17.=0 then 0 else 1) If #2.*#10.=1 then if s=0 then 1 else if t<3 then 2 else if t<5 then 4 else 5.
Explanation Frequency (ef1)	#21.-32.	If all of #21. - #32 = 99, then 5, else 1
Number of Explanations (ef2)	#21. - #32.	If ef1=5 then 99 else (if #21.=1 then 1 else 0)+(if #22.=1 then 1 else 0)+(if #23.=1 then 1 else 0)+(if #24.=1 or #26.=1 then 1 else 0)+(if #28. or #30.=1 then 1 else 0)

Summary by Characteristic of P3, P4, P5,P6, P13, P14

Characteristic	Coding Schema #'s	Logic of scoring
Algorithm	P3, P6, P13, P14	sum of algorithm scores/# of algorithm scores
Arithmetic/ Algebra	P5, P6, P13, P14	sum of algebra scores/# of algebra scores
Symbolic	P3, P4, P5, P13, P14	sum of symbolic scores/# of symbolic scores
Graph	P13, P14	sum of graph scores/# of graph scores
Correct Answer	P3, P4, P6	sum of correct answer scores/# of correct answer scores
Correct Solution	P3, P4, P5, P6	sum of correct solution scores/# of correct solution scores
Discussion	P3, P4	sum of discussion scores/# of discussion scores
Explanation Frequency (ef1)	P13, P14	sum of explanation frequency scores/# of explanation frequency scores
Number of Explanations (ef2)	P13, P14	sum of number of explanations scores/# of number of explanations scores
Conceptual	P3, P5, P6	sum of conceptual scores/#of conceptual scores

Calculus

and

Computer Supported Learning

Final Report

Appendix 4

Sample Maple Labs

Graphing Using Maple

Reference: Chapter P and Review Modules.

Team-Work to do in class and report is due at 5:45 p.m.:

For each the following functions: describe its **domain** and **range**; explain if it has any **symmetry** (even or odd properties); indicate the ***x*-intercept(s)** , the ***y*-intercept** and any **asymptotes**; and finally, provide your best sketch of each graph.

$f(x) = 16x^2 - 16x + 1$; $g(x) = 64x^4 - 16x^2 + 1$; $h(x) = 64x^3 - 16x^2 + 1$; $k(x) = x^2(8x-1)^2(2x-1)$;

$l(x) = \dfrac{1}{(8x-1)^2}$; $m(x) = \dfrac{1}{64x^4 - 16x^2 + 1}$; $n(x) = \dfrac{4x}{64x^4 - 16x^2 + 1}$

Bonus additional Work using technologies (graphic calculator, Maple or Live Math from the WebCal course). **This Team work is due in 1 week.**

For each function that you studied before: plot graphs with a good choice of *x* and *y*-window; confirm all results you previously predicted; read and state the coordinates of all "turning points".

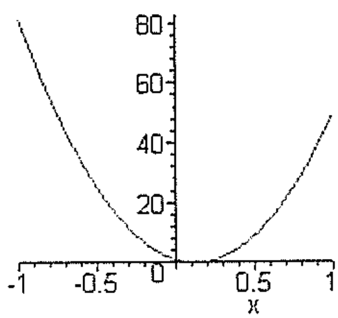
Some hints if you go to 5B3 to use Maple:

First log in as usual and double click the Maple icon. You will obtain a blank Maple worksheet.
Here is how to start:

```
with(student):
f:=64*x^2-16*x+1;
```

$$f := 64\,x^2 - 16\,x + 1$$

```
plot(f,x=-1..1);
```




```
solve(f=0,x);
```

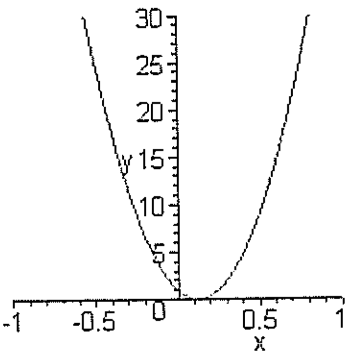
$$\frac{1}{8}, \frac{1}{8}$$

```
fsolve(f=0,x);
```

.1250000000, .1250000000

```
plot(f,x=-1..1,y=0..30);
```

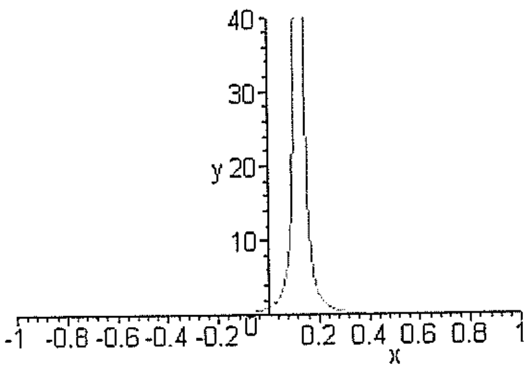
Graphing Using Maple



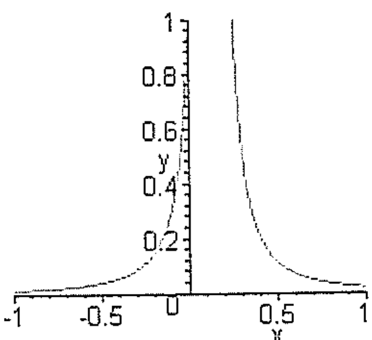
l:=1/(8*x-1)^2;

$$l := \frac{1}{(8\,x-1)^2}$$

plot(l,x=-1..1,y=0..40,discont=true);



plot(l,x=-1..1,y=0..1,discont=true);



Try to imitate these instructions and enter algebraic expressions as you have see it done in the above example. Note that you type in only after the Maple prompt >, and on the next line Maple will reply to you.

END

MAPLE LIMIT COMMANDS

There are two forms of the command: Limit (inert form), and limit (active form).

If you use the inert form, Limit, Maple just displays the limit in standard mathematical notation. This is what "inert" means: **no computation is carried out.**

If you use the active form, limit, Maple computes the value of the limit and displays it. Thus "active" means: **computation is carried out.**

The following examples display the syntax of these commands and a clever way of combining these two forms.

```
Limit(3*x+2,x=-1)=limit(3*x+2,x=-1);
```

$$\lim_{x \to (-1)} 3x + 2 = -1$$

```
Limit((7*x^3-4*x^2+2*x-5)/(x^3-3*x^2+x-3),x=2)=limit((7*x^3-4*x^2+2*x-5)/(x^3-3*x^2+x-3),x=2);
```

$$\lim_{x \to 2} \frac{7x^3 - 4x^2 + 2x - 5}{x^3 - 3x^2 + x - 3} = \frac{-39}{5}$$

The examples above could easily be calculated using our limit theorems.

How about some of the more interesting limits that we tackled by approximation and graphs in sections 2 and 3 above?

We note that all of these limits involved ratios of two functions, and in all cases the limit of the denominator was zero, hence the limit theorem about the quotient of two functions does not apply.

```
Limit(((x^2-1)*(x+1))/(2*(x-1)),x=1)=limit(((x^2-1)*(x+1))/(2*(x-1)),x=1);
```

$$\lim_{x \to 1} \frac{(x^2 - 1)(x + 1)}{2x - 2} = 2$$

```
Limit(sin(x)/x,x=0)=limit(sin(x)/x,x=0);
```

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

```
Limit(x*sin(1/x),x=0)=limit(x*sin(1/x),x=0);
```

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

The question that you might very well ask at this point is: if Maple will compute all of these limits for us so swiftly and painlessly, what is there for us to do?

The answer is not as simple as the question. A first attempt might be: when Maple computes a limit for us, it is in effect telling us something about the function and its graph, so we must be able to interpret Maple's computation and understand what it implies about the function and its graph.

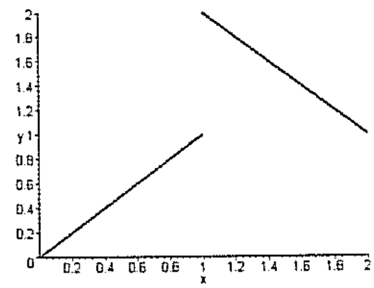
We begin this process in the next sections.

Section #6 : Half-Limits

We begin this section by looking at the graph of a function, $f(x)$.

f:=piecewise(0<x and x<1,x,1<=x and x<2,3-x); plot(f,x=0..2,y=0..2,color=black,thickness=3, discont=true);

$$f := \begin{cases} x & -x < 0 \text{ and } x-1<0 \\ 3-x & 1-x \le 0 \text{ and } x-2<0 \end{cases}$$



If someone asked us what the value of $\lim_{x\to 1} f(x)$ is, what should we say?

First we try Maple.
Limit(f,x=1)=limit(f,x=1);

$$\lim_{x \to 1} \begin{cases} x & -x < 0 \text{ and } x-1<0 \\ 3-x & 1-x \le 0 \text{ and } x-2<0 \end{cases} = \textit{undefined}$$

So Maple says the limit is undefined. Remember, our definition of the limit is **<u>a single number L</u>** that $f(x)$ gets close to when x gets close to a .

From the graph of this piecewise defined function $f(x)$ we can see that when x is less than 1 but getting closer and closer to 1, the values of $f(x)$ are getting closer and closer to 1. However, when x is greater than 1 but getting closer and closer to 1, the values of $f(x)$ are getting closer and closer to 2. Thus, we can see what number $f(x)$ is getting close to as x gets close to 1 depends upon which side of 1 the variable x is on. To be able to deal with this kind of situation we define what are called half-limits.

Limit from the left:

We write $\lim_{x\to a^-} f(x)$ and read it as "the limit of $f(x)$ as x approaches a from the left" to refer to **<u>a single number L</u>** that $f(x)$ gets close to when x gets close to a but is to the left of a , *i.e.* , $x < a$. We can ask Maple to write (inert form) or compute (active form) such limits by adding the word "left" to the command.

Limit(f,x=1,left)=limit(f,x=1,left);

$$\lim_{x \to 1-} \begin{cases} x & -x < 0 \text{ and } x-1<0 \\ 3-x & 1-x \le 0 \text{ and } x-2<0 \end{cases} = 1$$

Limit from the right:

We write $\lim_{x\to a^+} f(x)$ and read it as "the limit of $f(x)$ as x approaches a from the right" to refer to **a single number** L that $f(x)$ gets close to when x gets close to a but is to the right of a , i.e. , $a < x$. We can ask Maple to write (inert form) or compute (active form) such limits by adding the word "right" to the command.

Limit(f,x=1,right)=limit(f,x=1,right);

$$\lim_{x \to 1+} \begin{cases} x & -x < 0 \text{ and } x - 1 < 0 \\ 3 - x & 1 - x \le 0 \text{ and } x - 2 < 0 \end{cases} = 2$$

Connection between half-limits and the whole limit:
There is a Theorem that connects these ideas.
Theorem: The whole limit, $\lim_{x \to a} f(x)$, exists, and is L if and only if:

(1) the half limit from the left, $\lim_{x \to a^-} f(x)$, exists;

(2) the half limit from the right, $\lim_{x \to a^+} f(x)$, exists;

(3) the two half limits are equal, and equal to L, *i.e.*, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

Problem for <u>YOU</u> to do

Use Maple to plot the function $f(x) = \begin{cases} x\,\hat{}\,2-1 & , \quad x < -1 \\ 2 * x - 1 & , \quad -1 \le x \le 1 \\ 1 - x\,\hat{}\,2 & , \quad 1 < x \end{cases} = \begin{cases} x^2 - 1 & , \quad x < -1 \\ 2x - 1 & , \quad -1 \le x \le 1 \\ 1 - x^2 & , \quad 1 < x \end{cases}$ on the x-interval [-3,3].

Predict what the following half limits are:

$$\lim_{x \to -1^-} f(x) \quad \lim_{x \to -1^+} f(x) \quad \lim_{x \to -1} f(x)$$
$$\lim_{x \to 0^-} f(x) \quad \lim_{x \to 0^+} f(x) \quad \lim_{x \to 0} f(x)$$
$$\lim_{x \to 1^-} f(x) \quad \lim_{x \to 1^+} f(x) \quad \lim_{x \to 1} f(x)$$

Use Maple to compute each of these limits and see how good your predictions were.

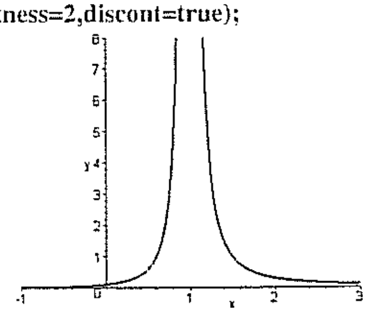
Section #7 : Infinite Limits

We begin this section by looking at another graph.

f:=(x^2-1)/(10*(x-1)^3);

$$f := \frac{1}{10} \frac{x^2 - 1}{(x-1)^3}$$

plot(f,x=-1..3,y=0..8,color=black,thickness=2,discont=true);



If someone were to ask you what is the value of $\lim_{x \to 1} f(x)$, what would you respond?

This does not fit our original definition using ϵ and δ, but a new definition can be made to cover this case. See me if you want to see this definition.

We should check our answer using Maple.

Limit(f,x=1)=limit(f,x=1);

$$\lim_{x \to 1} \frac{1}{10} \frac{x^2 - 1}{(x-1)^3} = \infty$$

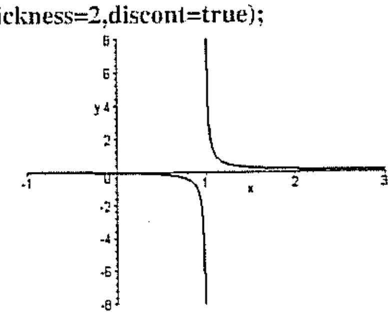
Hopefully you guessed in advance what this answer was going to be.
When the answer to a limit is infinite we refer to the limit as an "infinite limit".

Warning: Some textbooks refer to such limits as existing, but infinite. Other textbooks say that the limit does not exist, but that the limit diverges to infinity. Whatever words are used, the notation for this situation is the same: $\lim_{x \to a} f(x) = \infty$.

Now we examine another example.
f:=(x^2-1)/(20*(x-1)^2);

$$f := \frac{1}{20} \frac{x^2 - 1}{(x-1)^2}$$

plot(f,x=-1..3,y=-8..8,color=black,thickness=2,discont=true);



If someone were to ask you what is the value of $\lim_{x \to 1} f(x)$, what would you respond?

We note that the behaviour of the function is different on each side of $x = 1$, so perhaps half limits would be appropriate here.

We should check our answer using Maple.
Limit(f,x=1,left)=limit(f,x=1,left); Limit(f,x=1,right)=limit(f,x=1,right); Limit(f,x=1)=limit(f,x=1);

$$\lim_{x \to 1-} \frac{1}{20} \frac{x^2 - 1}{(x-1)^2} = -\infty$$

$$\lim_{x \to 1+} \frac{1}{20} \frac{x^2 - 1}{(x-1)^2} = \infty$$

$$\lim_{x \to 1} \frac{1}{20} \frac{x^2 - 1}{(x-1)^2} = \mathit{undefined}$$

We note that if we just asked Maple for the whole limit here we would not have as much information as the two half limits supply.

Note also that Maple seems to be following our Theorem concerning half limits, even for infinite limits. That is, because $-\infty \neq \infty$, Maple says that the whole limit is undefined. If we had asked Maple for half limits in the previous problem it would have said that they were both ∞, hence it reported the whole limit as ∞.

Problem for YOU to do

Plot the function $f(x)=\dfrac{x^3-1}{(x-1)(x+1)x^2}$.

From the graph predict the following limits:

$$\lim_{x\to -1^-} f(x) \quad \lim_{x\to -1^+} f(x) \quad \lim_{x\to -1} f(x)$$
$$\lim_{x\to 0^-} f(x) \quad \lim_{x\to 0^+} f(x) \quad \lim_{x\to 0} f(x)$$
$$\lim_{x\to 1^-} f(x) \quad \lim_{x\to 1^+} f(x) \quad \lim_{x\to 1} f(x)$$

Use Maple to compute the limits. How well did you predict the results?

Section #8 : Limits at Infinity

This is the last topic for today.

Sometimes, instead of asking what happens to values of $f(x)$ as x gets close to a number a, we want to ask about what happens when x approaches one or the other edge of the x-axis. In the language of limits, such limits are called limits at infinity, and written as $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to \infty} f(x)$ for the left and right edges respectively.

Note that our definition in terms of ϵ and δ does not work properly for this type of limit.

Any person wishing to see the appropriate definition for this type of limit can ask me about it.

We have already spent considerable time investigating the behaviour of functions at the edges of the x-axis, in particular for rational functions. We know from those efforts that functions either head towards $-\infty$, head towards ∞, approach a constant value c, or behave in a periodic fashion (like sin(x)). In the case where a function approaches a constant value c, we say the the function has a Horizontal Asymptote (HA), and is asymptotic to the line $y = c$. Thus, this type of limit provides us with symbolic notation and a way of describing such situations.

That is, $f(x)$ is asymptotic to $y = c$ at the left edge if and only if $\lim_{x\to -\infty} f(x) = c$.

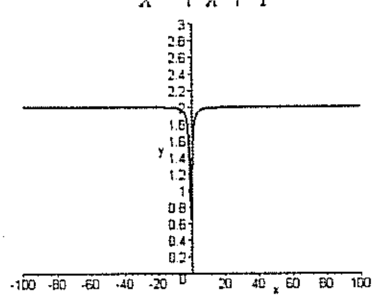
Similarly, f(x) is asymptotic to y = c at the right edge if and only if $\lim_{x\to \infty} f(x) = c$.

Suppose we look at an example.

We begin with a graph of the function.

f:=(2*x^2+2*x+1)/(x^2+x+1); plot(f,x=-100..100,y=0..3,color=black,thickness=2);

$$f := \frac{2x^2+2x+1}{x^2+x+1}$$



From the graph can you tell the values of the following limits: $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to \infty} f(x)$

Now we use Maple to compute the answers.

Limit(f,x=-infinity)=limit(f,x=-infinity); Limit(f,x=infinity)=limit(f,x=infinity);

$$\lim_{x \to (-\infty)} \frac{2x^2 + 2x + 1}{x^2 + x + 1} = 2$$

$$\lim_{x \to \infty} \frac{2x^2 + 2x + 1}{x^2 + x + 1} = 2$$

Lab Report <u>Warning:</u> Do not open this section until you have completed your journey through all of the sections above.

<u>Second Warning:</u> Do not open this section until you have completed your journey through all of the sections above.

<u>Third and Final Warning:</u> Do not open this section until you have completed your journey through all of the sections above. Problem for <u>YOU</u> to do

Plot the graph of $f(x) = \dfrac{3x^5 - 3x + 10}{-2x^6 + 3x^2 - 4}$

Based on the graph, predict the values of the following limits: $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$

Use Maple to calculate the limits and hopefully verify your predictions.

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Appendix 5

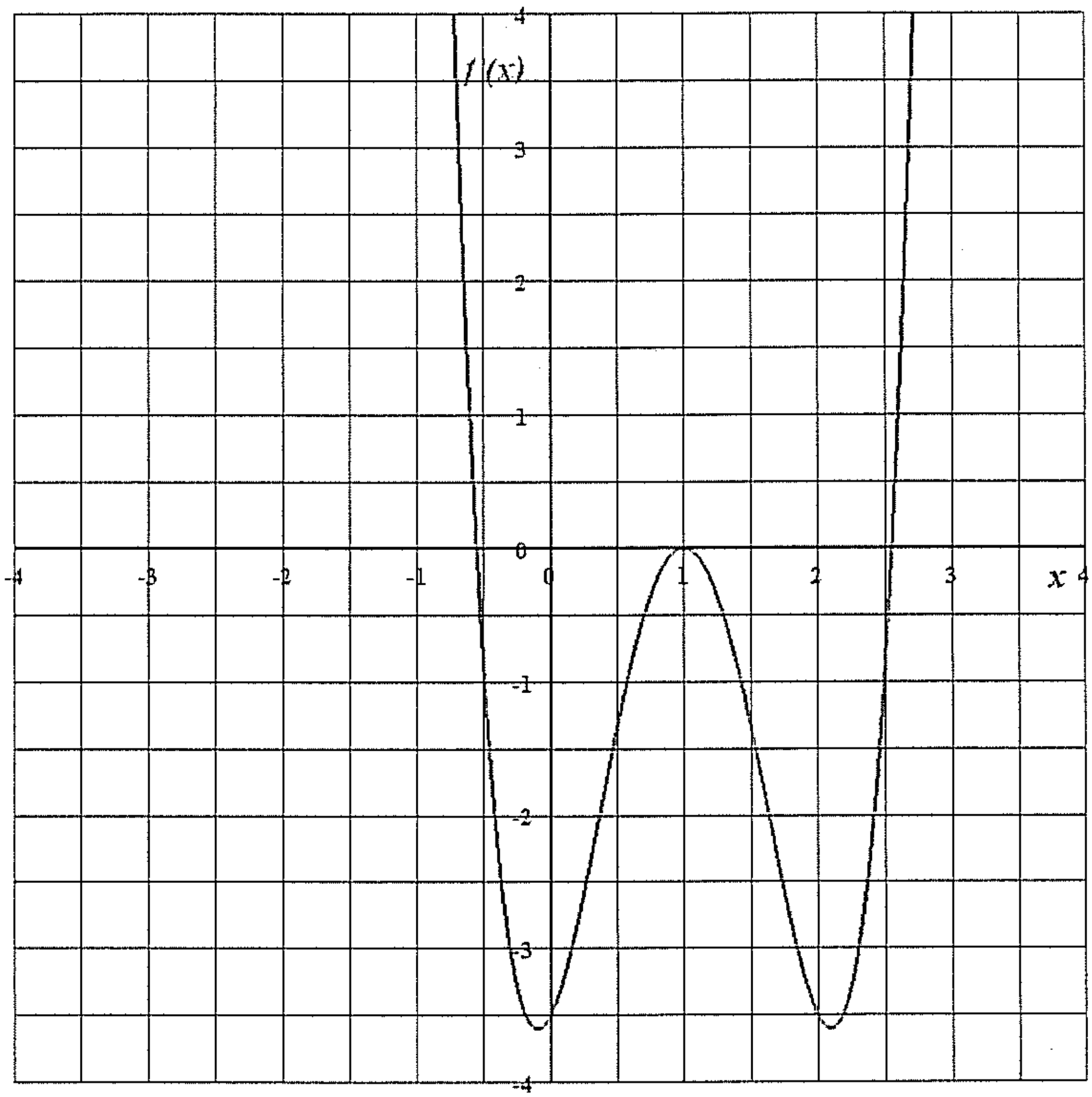
Feedback Quiz

Quiz

Name: ______________________________ Student #: _______________

Given the graph below of the function $f'(x)$, on the same set of axes sketch a reasoned graph of $f(x)$. Be sure to write clearly the steps of reasoning that you used to deduce the shape and features of your sketch. You may use the back of this sheet to write such explanations.

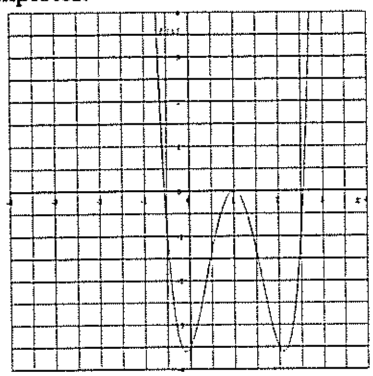
N.B. The graph below of $f'(x)$ is that of a polynomial so there are no asymptotes!



Quiz: Solution

Given the graph below of the function $f'(x)$, on the same set of axes sketch a reasoned graph of $f(x)$. Be sure to write clearly the steps of reasoning that you used to deduce the shape and features of your sketch. You may use the back of this sheet to write such explanations.

N.B. The graph below of $f'(x)$ is that of a polynomial so there are no asymptotes!



Solution:

N.B. The most common error is to forget that you are looking at the graph of f' and trying to deduce from it characteristics of f. The key is to remember that the direction (increasing/decreasing) of f is connected to the sign (+/-) of f' and that the concavity (up/down) of f is connected to the direction (increasing/decreasing) of f'.

Step 1: Locate the x-intercepts of f'.
We note that the x-intercepts of f' lie at approximately $x = -0.6$ and 2.6. These points are endpoints of intervals where f' may change sign.
Step 2: Locate intervals where the graph of f' lies above (below) the x-axis.
Looking at the graph of f' we note that on the x-intervals (-4,-0.6) and (2.6,4) the graph is above the x-axis, *i.e.*, the function f' is positive there. Similarly, we note that the graph of f' lies below the x-axis on the x-intervals (-0.6,1) and (1,2.6).

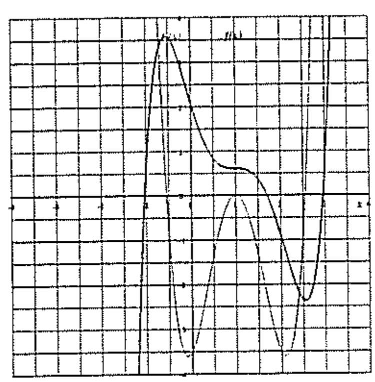
What do the observations from Step 1 and Step 2 tell us about the graph of f?
The graph of f is increasing on the x-interval (-4,-0.6), reaches a local maximum at $x = -0.6$, decreases on the x-interval (-0.6,1), reaches a stationary point at $x = 1$ (because even though f' is zero there it does not change sign, so f does not change direction as it does at a maximum or minimum), decreases on (1,2.6), reaches a minimum at $x = 2.6$, and increases on the x-interval (2.6,4).

Steps 3 & 4: Locate the local maxima/minima on the graph of f', and the x-intervals where f' has a single direction.
We note that, starting at the left edge, the graph of f' is decreasing, reaches a minimum value at $x = -0.15$, increases to a maximum value at $x = 1$, decreases to a minimum value at $x = 2.15$, and then increases to the right edge. That is, f' is decreasing on (-4,-0.15), has a minimum at $x = -0.15$, is increasing on (-0.15,1), has a maximum at $x = 1$, is decreasing on (1,2.15), has a minimum at $x = 2.15$, and then is increasing on (2.15,4).

What do the observations from Steps 3 & 4 tell us about the graph of f?
The graph of f is concave down on the x-interval (-4,-0.15), changes concavity at a point of inflection with x-coordinate of $x = -0.15$, is concave up on the x-interval (-0.15,1), changes concavity at a point of inflection with x-coordinate of $x = 1$, is concave down on the x-interval (1,2.15), changes concavity at a point of inflection with x-coordinate of $x = 2.15$, is concave up on the x-interval (2.15,4).

To make it easier to see the graph of f that we are preparing we enter the information that we determined above in a table.

x	-4		-1.6		-0.15		1		2.15		2.6		4
f		↗	‒ M	↘	↘	↘	SP ‒	↘	↘	↘	m ‒	↗	
		⌢	⌢	⌢	PI	⌣	PI	⌢	PI	⌣	⌣	⌣	
f'		+	0	-	-	-	0	-	-	-	0	+	
		↘	↘	↘	m	↗	M	↘	m	↗	↗	↗	



Note that we do not actually have any values of f, thus we do not know any y coordinates. In Calculus II (or in Physics) we learn how to estimate these values. For now, it is only the rough shape that we are worried about.

Coding Schema

1. Number of left to right passes through the graph of f'
   1. Single pass
   2. Two passes
   3. Organizational principle unclear

2. Follow general pattern of looking at sign of f' first and then direction of f'
   1. Yes
   2. Clearly not
   3. Unclear

3. Presence/absence of graph
   1. Complete Graph Present
   2. Incomplete Graph Present
   99. Absent

<u>If 3. above is coded as 99, then code 4 - 11 & 20 as 99.</u>

4. For each zero of f', is an Extrema/Stationary Point of f drawn on the graph: Count # of omissions

5. For each zero of f', is the correct Extrema/Stationary Point of f drawn on the graph: Count # of errors or omissions

6. Pattern on graph of intervals of increase and decrease of f: Count # of errors or omissions

7. For each change in direction of f', is a Point of Inflection of f drawn: Count # of errors or omissions

8. For each change in direction of f', is the correct Point of inflection of f drawn on the graph: Count # of errors or omissions

9. Pattern of intervals of concavity on graph of f: Count # of errors or omissions

10. Drawing of stationary point:
    1. drawn with zero slope and change of concavity
    2. drawn with one of change of concavity but not zero slope
    3. drawn with zero slope but not with change of concavity (*i.e.*, drawn as extremum)
    4. nothing special drawn at this point

11. Verbal Explanation of relationship between sign (-/+) of f' and direction (↘/↗) of f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

12. Verbal Explanation of relationship between zeroes (0) of f' and extrema or stationary points of f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

13. Verbal Explanation of relationship between direction (↘/↗) of f' and concavity (⌒/⌣) of f
    1. only correct statement(s)
    2. mixture of correct and uncorrect statements
    3. only false statement(s)
    99. no statement

14. Verbal Explanation of relationship between extrema (Max./min) of f' and points of inflection of f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

15. Presence/absence of table
    1. Complete Table Present
    2. Incomplete Table Present
    99. Absent

<u>If 15. above is coded as 99, then 16 - 20 below are all coded as 99.</u>

16. Tabular Explanation of relationship between sign (-/+) of f' and direction (↘/↗) of f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

17. Tabular Explanation of relationship between zeroes (0) of f' and extrema or stationary points of f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

18. Tabular Explanation of relationship between direction (↘/↗) of f' and concavity (⌢/⌣) of f
    1. only correct statement(s)
    2. mixture of correct and uncorrect statements
    3. only false statement(s)
    99. no statement

19. Tabular Explanation of relationship between extrema (Max./min) of f' and points of inflection of f
    1. only correct statement(s)
    2. mixture of correct and incorrect statements
    3. only false statement(s)
    99. no statement

20. Number of discrepancies between table and graph - count of errors or omissions (note if either of 3. or 17. is coded as 99, then this is automatically coded as 99).

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Appendix 6

WebCal Worksheets

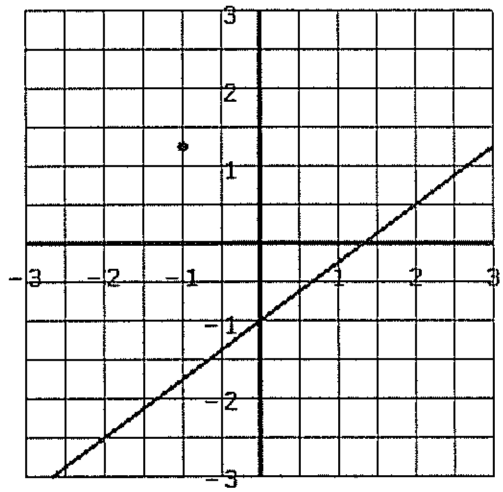
Name: ______________________ Name: ______________________

Name: ______________________ Name: ______________________

In the problems below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of the **previous** sheet.

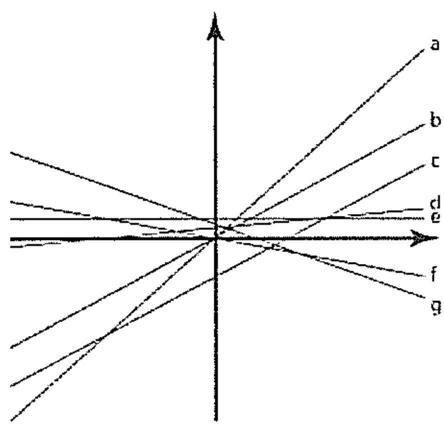
1. Given the graph below with a line and a point on the 3×3 grid:
   a. Draw a line through the given point and parallel to the given line.
   b. Use the **point-slope formula** for a straight line to write an equation for the line that you drew.
   c. Use the **point-slope formula** for a straight line to write an equation for the line through the given point but **perpendicular** to the given line.

Solutions:

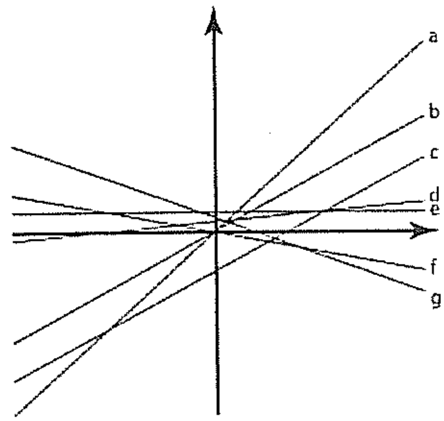


2. The graph on the right shows several straight lines:
   a. Which lines have positive and which lines have negative slopes?
   b. Rank these lines from the steepest to the least steep.
   c. Select any pair of lines that have the same slope.
   d. Select any and all lines which represent proportional relationships between the dependent and independent variables.

Solutions:



3. The graph on the right shows several straight lines. Wherever possible match a line with one of the equations below. **Note that some equations may not match any lines and some lines may not match any equations.**



   a. $\frac{y+1}{x-1} = \frac{y-5}{x+1}$ matches line ____ because

   b. $y = 3$ matches line ____ because

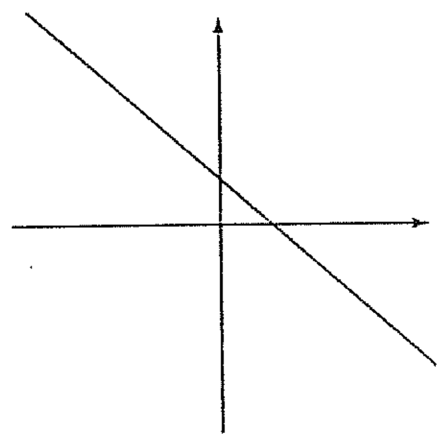
   c. $x = -6t - 2$; $y = 2t + 3$ matches line ____ because

   d. $y = 4x$ matches line ____ because

   e. $3x - 5y + 4 = 0$ matches line ____ because

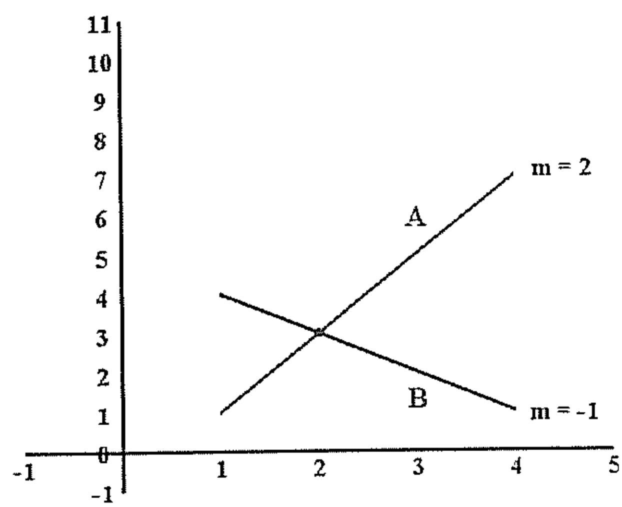
   f. $\frac{y-3}{x-4} = 2$ matches line ____ because

4. The graph on the right shows the line $y = mx + b$. Sketch each of the lines given below on the same grid and be sure to identify each line (colour code or label). Explain your reasoning clearly.
   a. $y = mx - b$
   b. $y = -mx + b$
   c. $y = -mx - b$



5. The graph on the right shows line segments A and B.
   a. Derive an equation for the line of which A is a segment. Show all work.
   b. Determine the height on line segment B when $x = 3$. Show all work.

Solutions:

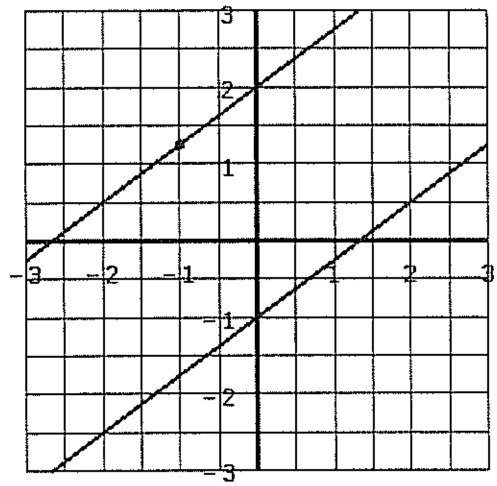


In the problems below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of the **previous** sheet.

1. Given the graph below with a line and a point on the 3×3 grid:
   a. Draw a line through the given point and parallel to the given line.
   b. Use the **point-slope formula** for a straight line to write an equation for the line that you drew.
   c. Use the **point-slope formula** for a straight line to write an equation for the line through the given point but **perpendicular** to the given line.

Solutions:

b. We note that the given point has coordinates (-1,1.25).
We note that parallel lines have the same slope, and the given line passes through the grid points (0,-1) and (2,½), hence has slope $m = \frac{(\frac{1}{2}-(-1))}{2-0} = \frac{\frac{1}{2}+1}{2} = \frac{\frac{3}{2}}{2} = \frac{3}{4}$.



Thus, the line we drew has equation

$$\frac{y-1.25}{x-(-1)} = \frac{3}{4} \quad \text{or} \quad \frac{y-1.25}{x+1} = \frac{3}{4}$$

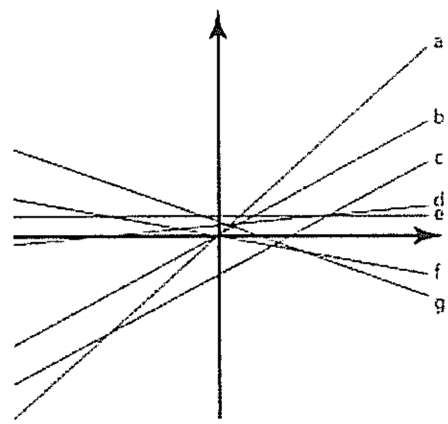
Note that there is no need at this time to rewrite the equation in slope-y-intercept form.

c. We note that if two lines are perpendicular, then the product of their slopes is -1. Thus, an equation for a line perpendicular to the two lines already drawn will have slope m, where $m \times \frac{3}{4} = -1$ or $m = -\frac{4}{3}$

We know that the perpendicular line that we seek passes through (-1,1.25).

Thus, the line we seek has equation $\frac{y-1.25}{x-(-1)} = -\frac{4}{3}$ or $\frac{y-1.25}{x+1} = -\frac{4}{3}$

2. The graph on the right shows several straight lines:
   a. Which lines have positive and which lines have negative slopes?
   b. Rank these lines from the steepest to the least steep.
   c. Select any pair of lines that have the same slope.
   d. Select any and all lines which represent proportional relationships between the dependent and independent variables.



Solutions:

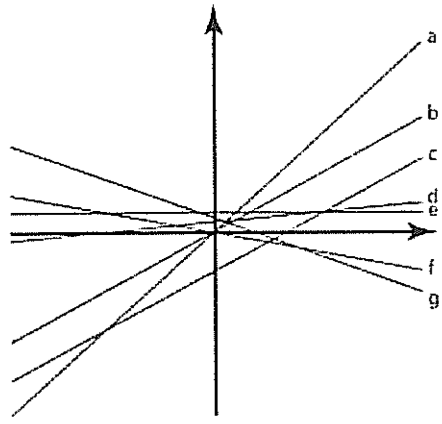
a. Lines with positive slope rise (increase in y-value) as you move from left to right along them. That is, as x gets larger (left to right), y gets larger (bottom to top). Lines with negative slope fall (decrease in y-value) as you move from left to right along them. That is, x gets larger (left to right), y gets smaller (top to bottom). It is important to notice the convention that when we discuss increasing versus decreasing, we only discuss it in left to right terms. If we did not have this convention, then any line (other than horizontal ones) could seem to be increasing, merely by choice of direction, left to right or right to left.
Following the convention outline above: a, b, c, d are increasing, hence have positive slopes; f and g are decreasing, hence have negative slopes; e appears to be horizontal, hence has zero as its slope.

b. Given that there is no scale on the graph this is not so simple to do. We note that the larger the absolute value of the slope, the steeper the line. That is, we do not care about the sign of the slope, just its magnitude or size, when measuring steepness. It seems reasonably clear that a is the steepest line, b appears ever so slightly steeper than g, and since c is parallel, it too is a touch steeper than g, then g, followed by f, which is a touch steeper than d, with the horizontal line e being the least steep. To give this in list form, a, {b-c}, g, f, d, e.

c. Only parallel line have the same slope, and the only parallel lines are b and c.

d. Two variables, say y and x, are proportional if $y = kx$, where k is some constant, called the constant of proportionality. We note that this means that the point (0,0), *i.e.*, the origin, lies on all graphs of proportional relationships. The only lines satisfying this condition are a, b and f.

3. The graph on the right shows several straight lines. Wherever possible match a line with one of the equations below. **Note that some equations may not match any lines and some lines may not match any equations.**



a. $\frac{y+1}{x-1}=\frac{y-5}{x+1}$ matches line ?. We note that there is no scale on either the x-axis or the y-axis. This makes the task quite difficult because the obvious clue from the given equation it that the line passes through the points (1,-1) and (-1,5), and hence has a slope of $\frac{5-(-1)}{-1-1}=\frac{6}{-2}=-3$. Further, this line does not pass through the origin, because $-1=\frac{0+1}{0-1}\neq\frac{0-5}{0+1}=-5$. Thus, the only line that could possibly match the given equation would be g.

b. $y = 3$ matches line e, if it matches any line, because $y = 3$ is the equation of a horizontal line and e is the only horizontal line. Of course, whether it matches or not depends upon the y-axis scale, which was not given.

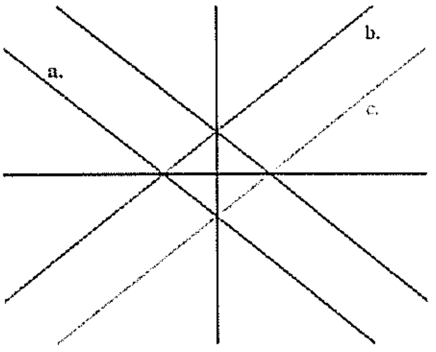
c. $x = -6t - 2$; $y = 2t + 3$: from the parametric equations we see that this line has slope $\frac{2}{-6}=-\frac{1}{3}$, and passes through the point (-2,3) when $t = 0$. There are only two lines with negative slope, f and g. To check if the parametric equation describes a line passing through the origin, we solve two equations for t, created by substituting $x = 0$ and $y = 0$: $0 = -6t - 2 \Rightarrow 6t = -2 \Rightarrow t = -1/3$ and $0 = 2t + 3 \Rightarrow -2t = 3 \Rightarrow t = -3/2$. Since these two values are not the same, the parametric line does not pass through the origin. This means that the only line that this equation could match would be g, but this is not the case if the answer to b. or a. is correct.

d. $y = 4x$: this equation describes a positively sloped line that passes through the origin, hence it can only match a or b. Without a scale on either axis we cannot know if it truly matches either of these or not.

e. $3x - 5y + 4 = 0$: rewriting this equation we obtain $5y = 3x + 4 \Rightarrow y = (3/5)x + (4/5)$, a line with positive slope and a positive y intercept. Without scales on the axes we cannot know for sure, but d is the only possible line to match this equation.

f. $\frac{y-3}{x-4}=2$: this equation describes a line with a positive slope of 2, passing through the point (4,3). We can also see, by substituting (0,0) into the equation, that the line does not pass through the origin. This leaves two candidates, c and d, both with positive slope and not passing through the origin. We note that the y-intercept of the given line is: $\frac{y-3}{0-4}=2 \Leftrightarrow y-3=2(-4) \Leftrightarrow y=-5$. We also note that the x-intercept of the given line is: $\frac{0-3}{x-4}=2 \Leftrightarrow -3=2x-8 \Leftrightarrow 2x=5 \Leftrightarrow x=5/2$. Only line d matches both of these two criteria, a negative y-intercept and a positive x-intercept, thus if any line matches it is d.

4. The graph on the right shows the line $y = mx + b$. Sketch each of the lines given below on the same grid and be sure to identify each line (colour code or label). Explain your reasoning clearly.
   a. $y = mx - b$
   b. $y = -mx + b$
   c. $y = -mx - b$



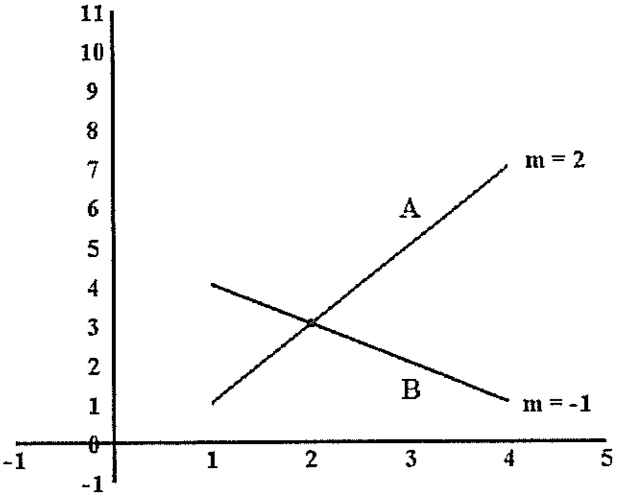
Solution:

a. From the given line we observe that the slope, m, is clearly negative, and the y-intercept, b, is clearly positive. Thus, the line in a. should be parallel to the given line (same slope), and with negative y-intercept, but of the same absolute value as the given line.

b. This line passes through the same y-intercept, but has a slope of the same absolute value, but opposite in sign, hence positive. Such a line should be symmetric to the original line through the mirror of the y-axis.

c. This line has the same y-intercept as the line in a., but has the slope of the line in b. In this case, the line and the original line will be symmetric about the x-axis.

5. The graph on the right shows line segments A and B.
   a. Derive an equation for the line of which A is a segment. Show all work.
   b. Determine the height on line segment B when $x = 3$. Show all work.



Solutions:

a. We have been given the slope, $m = 2$, and we note that the line passes through the point (2,3). Thus, the line has equation: $\frac{y-3}{x-2} = 2$.

b. Just as in the previous case, we know the slope, $m = -1$ and a point (2,3). Thus, an equation for this line is: $\frac{y-3}{x-2} = -1$. Substituting in $x = 3$ we obtain:

$$\frac{y-3}{3-2} = -1 \Leftrightarrow y - 3 = -1 \Leftrightarrow y = 3 - 1 \Leftrightarrow y = 2$$

Alternatively, we can just note that since the slope is -1, and slope is the ratio of "rise" to "run". To go from $x = 2$ to $x = 3$, we need a "run" of 1, hence a "rise" of -1, so the y-value would be 3-1 = 2.

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You are sitting in the Science Centre, doing Calculus homework, and your best friend from High School, now a student in Social Science, comes over and starts reading over your shoulder. The question that you are reading says:

a. Given $f(x) = x^2$ and $g(x) = x^4$, both have "U" shaped graphs. If they are drawn on the same set of axes, how can you tell their graphs apart?

Your friend says "how can you tell the difference". In a short paragraph, writing complete sentences, and using the functions in a. above to illustrate, explain to your friend how we distinguish between the graphs of these two even-powered power functions.

The next question that you read says,

b. No power function with positive exponent has any asymptotes, but all power functions with negative exponents have both vertical asymptotes and horizontal asymptotes. Explain where (this means in terms of x-values) and why a power function like $h(x) = x^{-3}$ has: (i) a vertical asymptote; (ii) a horizontal asymptote.

Your friend watches you and then asks "what does "has a vertical asymptote" mean, and what does "has a horizontal asymptote" mean. Answer your friends questions first, then answer b. as well, all in a short paragraph, using complete sentences.

The next question that you read says,

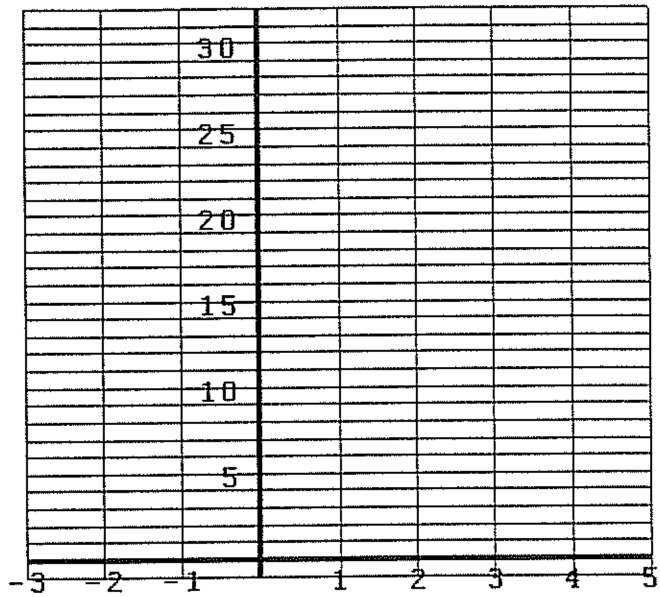
c. Given $f(x) = 2^x$ and $g(x) = x^2$, one is a power function and the other is an exponential function. Which is the power function and which is the exponential?

Your friend says "how can you tell the difference". In a short paragraph, writing complete sentences, and using the given functions to illustrate, explain to your friend how we distinguish between a formula for a power function and one for an exponential function.

The next question that you read, a continuation of c. above says,

d. Use the grid on the right to sketch graphs of both $f(x)$ and $g(x)$. (Distinguish between the two by using two different colours, or dashed and solid lines, explaining which function corresponds to which curve.) Which function dominates at the right edge?

Your friend watches you draw your sketches, and then asks "what does dominates at the right edge mean"? Draw your sketches as per instructions in b. above, and then write a short paragraph, using complete sentences, referring to your sketches as an illustration, to explain what is meant by "dominates at the right edge".



The last question that you read says,

e. Some exponential functions are referred to as "growth" and others as "decay". Give an example of a formula for each kind.

Your friend watches you write your two examples and sees you label one as growth and the other as decay, and then asks "when you look at a formula of an exponential function, how do you decide which is growth and which is decay, and what do the words growth and decay mean here". Give your examples as per instructions in e. above, and then write a short paragraph, using complete sentences, referring to your examples as an illustration, to answer your friend's questions.

growth function example: ____________________ decay function example: ____________________

You are sitting in the Science Centre, doing Calculus homework, and your best friend from High School, now a student in Social Science, comes over and starts reading over your shoulder. The question that you are reading says:

a. Given $f(x) = x^2$ and $g(x) = x^4$, both have "U" shaped graphs. If they are drawn on the same set of axes, how can you tell their graphs apart?

Your friend says "how can you tell the difference". In a short paragraph, writing complete sentences, and using the functions in a. above to illustrate, explain to your friend how we distinguish between the graphs of these two even-powered power functions.

Solution:

We know that the graph of a power function with an even positive integer power is "U" shaped. However, we also know that the higher the power, the higher the value of the function when $|x| > 1$. That is, for a given vertical line (*i.e.*, a particular value of x), where the vertical line lies either to the left of $x = -1$ or to the right of $x = 1$, the vertical line intersects the power function of lower power first, and then the other. For example, if $x = -2$, then $f(-2) = (-2)^2 = 4 < g(-2) = (-2)^4 = 16$. We usually say that the power function of higher power is "steeper" at the edges than the other. However, we also notice that for $-1 < x < 1$ (or $|x| < 1$), the power function with lower power is higher. For example, $f(½) = (½)^2 = ¼ > g(½) = (½)^4 = 1/16$.

The next question that you read says,

b. No power function with positive exponent has any asymptotes, but all power functions with negative exponents have both vertical asymptotes and horizontal asymptotes. Explain where (this means in terms of x-values) and why a power function like $h(x) = x^{-3}$ has: (i) a vertical asymptote; (ii) a horizontal asymptote.

Your friend watches you and then asks "what does "has a vertical asymptote" mean, and what does "has a horizontal asymptote" mean. Answer your friends questions first, then answer b. as well, all in a short paragraph, using complete sentences.

Solution:

A function $f(x)$ is said to have a vertical asymptote at $x = a$ if, as x values get closer and closer to a, the $f(x)$ values get larger and larger in absolute value, without any "bound" or number beyond which they do not grow. On the graph, this means that as x gets closer and closer to a, the graph gets steeper and steeper, looking more and more like a vertical line, in particular, the vertical line $x = a$, without actually touching the vertical line $x = a$. Thus, a vertical asymptote is a feature of a graph in the "middle" part of the graph, because it occurs at an actual value of x.

A function $f(x)$ is said to have a horizontal asymptote <u>at the edge of the graph</u> if, as x values either approach $-\infty$ or ∞ (the left and right edges respectively), then $f(x)$ values approach some constant value, c. The horizontal line $y = c$ is said to be the horizontal asymptote, but the asymptote occurs at an edge of the graph. Looking at the graph, we would notice that at an edge, the graph is becoming more and more like a horizontal line, in particular, the horizontal line $y = c$.

Power functions of the form Cx^r, where $r < 0$, have a vertical asymptote at $x = 0$ (the y-axis), and a horizontal asymptote, $y = 0$ (the x-axis), at both edges. The reasoning for these conclusions is as follows: when x is close to 0, then $|x|$ is small, hence $1/|x|$ is large. Thus, the closer that x gets to 0, the closer $x^r = 1/x^{-r}$ (where $r < 0$ so $-r > 0$) gets to infinity. Similarly, when $|x|$ gets very large (at either edge of the x-axis), then $1/|x|$ gets very small, *i.e.*, close to 0, and so does $x^r = 1/x^{-r}$.

Numerically we can examine the following tables of numbers, using $h(x) = x^{-3}$, to see these phenomena in action:

x	-0.1		-0.01		-0.001		0		0.001		0.01		0.1
$h(x)$	-10^3		-10^6		-10^9		undefined		10^9		10^6		10^3

We notice that in the table above, as x values get close to 0, but are negative, the $h(x)$ values head towards $-\infty$. Similarly, as x values get close to 0, but are positive, the $h(x)$ values head towards ∞. This pattern shows us that $h(x)$ has a vertical asymptote at $x = 0$.

x	-10^9		-10^6		-10^3		0		10^3		10^6		10^9
$h(x)$	-10^{-27}		-10^{-18}		-10^{-9}		undefined		10^{-9}		10^{-18}		10^{-27}

We notice that in the table above, as x values get closer and closer to either $-\infty$ or ∞, the $h(x)$ values seem to get closer and closer to 0. This pattern shows us that $h(x)$ has a horizontal asymptote at both edges.

The next question that you read says,

c. Given $f(x) = 2^x$ and $g(x) = x^2$, one is a power function and the other is an exponential function. Which is the power function and which is the exponential?

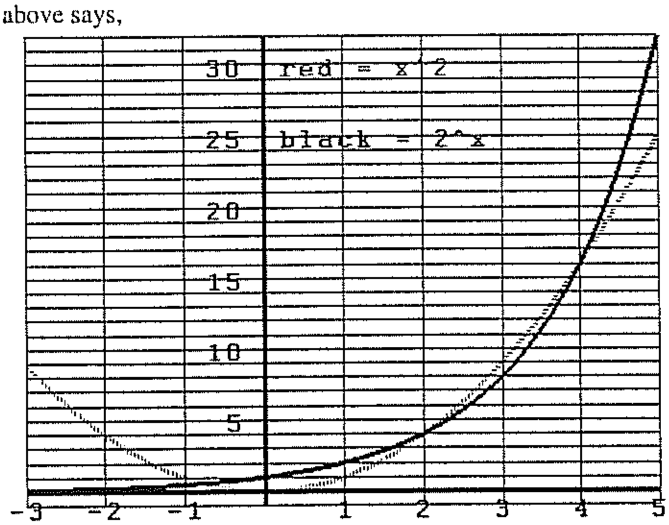
Your friend says "how can you tell the difference". In a short paragraph, writing complete sentences, and using the given functions to illustrate, explain to your friend how we distinguish between a formula for a power function and one for an exponential function.

Solution:

Both types of functions have the form coefficient (base)$^{\text{exponent}}$. A power function has a variable value for the base, but a constant value for the exponent. Thus, $g(x) = x^2$, with the variable x as base, and the constant 2 as exponent, is an example of a power function. An exponential function has a constant value for the base, but a variable value for the exponent. Thus, $f(x) = 2^x$, with the constant 2 as base, and the variable x as exponent, is an example of an exponential function. Note that there are two other possibilities: we could have a function where both the base and the exponent are constants, *e.g.*, $h(x) = 2^3$, but really this is just a constant function, namely $h(x) = 8$; we could also have a function where both the base and the exponent are variable, *e.g.*, $j(x) = x^{\sin(x)}$, this is neither an exponential nor a power function. Later in the course we will have to devise a special method just to deal with nasty functions like this one.

The next question that you read, a continuation of c. above says,

d. Use the grid on the right to sketch graphs of both $f(x)$ and $g(x)$. (Distinguish between the two by using two different colours, or dashed and solid lines, explaining which function corresponds to which curve.) Which function dominates at the right edge?



Your friend watches you draw your sketches, and then asks "what does dominates at the right edge mean"? Draw your sketches as per instructions in b. above, and then write a short paragraph, using complete sentences, referring to your sketches as an illustration, to explain what is meant by "dominates at the right edge".

Solution:

We say that one function, in this case $f(x)$, dominates another function, in this case $g(x)$, at the right edge of the graph, if, when the values of x get larger and larger (this is what is meant by the right edge), the values of the first function (in this case 2^x) are increasingly larger than those of the second function (in this case x^2). In a graph, a function having a larger value than another means that, for a fixed value of x, *i.e.*, on a vertical line, one function intersects the vertical line at a higher y value. We can see that even though the functions in this example reverse position twice, at the right edge of the graph 2^x is above x^2, and the gap between them is growing larger as we move further to the right.

The last question that you read says,

e. Some exponential functions are referred to as "growth" and others as "decay". Give an example of a formula for each kind.

Your friend watches you write your two examples and sees you label one as growth and the other as decay, and then asks "when you look at a formula of an exponential function, how do you decide which is growth and which is decay, and what do the words growth and decay mean here". Give your examples as per instructions in e. above, and then write a short paragraph, using complete sentences, referring to your examples as an illustration, to answer your friend's questions.

Solution:

growth function example = 2^x **decay function example = $(\frac{1}{2})^x$**

There are two types of exponential function graphs. One type has increasingly larger values of y as x increases (the graph heads upwards as we move our eyes left to right). The other type has decreasing values of y as x increases (the graph heads downwards as we move our eyes left to right). Increasing exponential functions are referred to as "growth" since they are often used to model such things as population growth. Decreasing exponential functions are referred to as "decay" since they are often used to model such things as radioactive decay. Both types of exponential functions have the form coefficient (constant base)$^{\text{variable exponent}}$,

but growth functions have a constant base which is larger than 1, while decay functions have a constant base which is smaller than 1 (but larger than 0 since it is difficult to understand what would be meant by raising a negative base to non-integer exponent values). Thus, since $2 > 1$, we can use 2^x as an example of a growth function. Similarly, since $0 < \frac{1}{2} < 1$, we can use $(\frac{1}{2})^x$ is a decay function.

Note: 1) In the above we are assuming that the "coefficient" is positive. Should the coefficient be negative, the graphs described above would be flipped about the x-axis, and would generally no longer work as models for population growth or radioactive decay.

ii. There is an alternative method of writing exponential functions. While this is not the appropriate place for a complete treatment of it, what follows is a brief discussion. We can say that all exponential functions can be written in the form: coefficient (base e)$^{\text{constant}\times x}$, usually this is written as Ce^{kx}.

(1) What is so special about base e that we should fix upon it? Mathematicians like to "joke" that physicists (sound levels, seismic disturbances, etc.) and chemists (pH) use base 10 because they count on their fingers and toes, and biologists (population growth) use base 2 because they count on their hands and feet. The truth is that base 10 is pretty much built into humans, not just our fingers and toes, but all our senses. That is, a sound has to be 10 times as loud before we can really perceive the difference, a star has to be 10 times as bright before we can see that it is brighter, etc. In defence of biologists, base 2 really does make it easier to calculate such things as doubling time in a population. Why then do mathematicians use base e? It turns out, and I hate to do this, that about half-way through the term we will know precisely why. For now, all I can say is that calculations done in Calculus are much cleaner and simpler if we use base e, hence we will do this.

(2) How does e^{kx} compare to a^x? How much you understand of the following discussion will depend upon what you already know, but do not panic since we will be reviewing or learning about this topic later. Since mathematicians are hooked on base e for exponential functions, they also use base e for logarithms, which are the inverse functions for exponentials (more about this in a week or two). In fact, mathematicians use base e logarithms so much that instead of writing $\log_e(x)$ they write $\ln(x)$, where ln (pronounced like "lawn") is short for either "logarithme naturelle" or "Naperian logarithm" (after J. J. Napier who was the first to really use base e). Hopefully you remember about the relationship between inverse functions (if not, fear not, we will cover it soon). The key is the notion that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, which in the case of e^x and $\ln(x)$ gives us the following two identities: $e^{\ln(x)} = x$ and $\ln(e^x) = x$. You may find these more familiar as $10^{\log(x)} = x$ and $\log(10^x) = x$. Using the first identity, $e^{\ln(x)} = x$, and replacing x with a, we obtain $e^{\ln(a)} = a$. Thus, $a^x = (e^{\ln(a)})^x = e^{\ln(a)x}$, so that we can use a^x interchangeably with e^{kx}, with $k = \ln(a)$.

(3) We know that when we represent the family of exponential functions as being of the form Ca^x, we talk of two subfamilies, decay functions, where $0 < a < 1$, and growth functions, where $1 < a$. What is the corresponding classification when we represent exponential function as being of the form Ce^{kx}? First we note that from above $k = \ln(a)$. Second we note that when $a = 1$, then $k = \ln(1) = 0$. Thus, $0 < a < 1$ is equivalent to saying that $-\infty < k = \ln(a) < \ln(1) = 0$, and $1 < a$ is equivalent to saying that $0 = \ln(1) < \ln(a) < k$. Thus decay corresponds to $k < 0$ and growth corresponds to $0 < k$.

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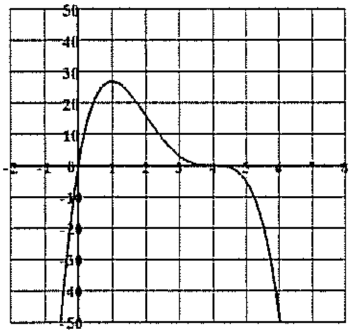
In the problems below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of this sheet.

1. Given the polynomial function $f(x) = x\,(x - 1)\,(x - 2)^3\,(x + 1)^3\,(x + 3)^4$:
   a. **Without** multiplying this out and placing it in standard form, determine the degree, leading coefficient and constant term of $f(x)$.
   b. Determine all the zeroes of this function, hence all of the x-intercepts of a graph of this function. Determine the "shape" of the graph "nearby" each x-intercept.
   c. **Without** computing a table of values for the function, **sketch** a graph of $f(x)$, explaining how it accurately portrays all information gleaned in a. and b. above.

Solutions:

2. Given the graph of a polynomial function g on the right, deduce a possible formula for g. Explain your reasoning, step by step.



Solutions:

3. Given the function $h(x) = -4x^5 - 20x^4 - 9x^3 + 106x^2 + 212x + 120$:
   a. Determine the x-intercepts of $h(x)$.
   b. **Without** computing a table of values for the function, **sketch** a graph of $h(x)$, explaining how it accurately portrays all information gleaned in a. and b. above.

Solutions:

In the problems below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of this sheet.

1. Given the polynomial function $f(x) = x(x-1)(x-2)^3(x+1)^3(x+3)^4$:
   a. **Without** multiplying this out and placing it in standard form, determine the degree, leading coefficient and constant term of $f(x)$.
   b. Determine all the zeroes of this function, hence all of the x-intercepts of a graph of this function. Determine the "shape" of the graph "nearby" each x-intercept.
   c. **Without** computing a table of values for the function, **sketch** a graph of $f(x)$, explaining how it accurately portrays all information gleaned in a. and b. above.

Solutions:

a. We need only add up the multiplicity of each linear factor to determine the degree of the polynomial. Thus, degree(f) = 1+1+3+3+4 = 12. That is, if we were to multiply this factored polynomial, then the highest power of x obtained with non-zero coefficient would be 12. We note that to determine the leading coefficient we need only determine the product, using multiplicity, of the leading coefficients of all factors. Thus, in this example the leading coefficient is $1\times1\times1^3\times1^3\times1^4 = 1$. Finally, the constant term is the same as the value of the y-intercept, which we obtain quite simply by substituting in $x = 0$: $f(0) = 0\,(0-1)(0-2)^3(0+1)^3(0+3)^4 = 0$. that is, this graph passes through the origin.

b. By the Factor and Remainder Theorems, $x = a$ is a zero of $f(x)$ (*i.e.*, $f(a) = 0$) if and only if $(x - a)$ is a factor of $f(x)$. Since we can see all of the factors in the form that $f(x)$ was given, then the zeroes of $f(x)$ are $x = 0, 1, 2, -1, -3$, or rearranged in left to right x-axis order, $x = -3, -1, 0, 1, 2$. The shape of the graph at each x-intercept is determined by the multiplicity of the factor.

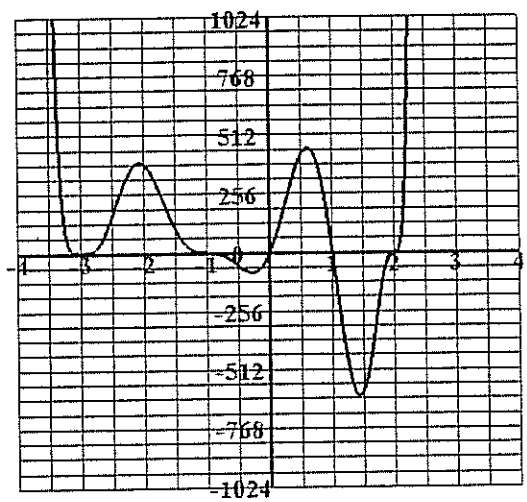
At $x = -3$, which corresponds to the factor $(x + 3)$, which has multiplicity 4, the graph will not cut through the x-axis, but instead bounce back in a "U" shape.

At $x = -1$, which corresponds to the factor $(x + 1)$, which has multiplicity 3, the graph will cut through the x-axis, but it will do it in the shape of the x-intercept of the function x^3.

At $x = 0$, which corresponds to the factor x, which has multiplicity 1, the graph will cut through the x-axis, but it will do it like an oblique line.

At $x = 1$, which corresponds to the factor $(x - 1)$, which has multiplicity 1, the graph will cut through the x-axis, but it will do it like an oblique line.

At $x = 2$, which corresponds to the factor $(x - 2)$, which has multiplicity 3, the graph will cut through the x-axis, but it will do it in the shape of the x-intercept of the function x^3.



c. We begin by noting that the leading term, which determines the behaviour of $f(x)$ at the edges, is just $1\times x^{12}$. Since the degree is even, at both edges the polynomial function has the same behaviour, *i.e.*, either it goes up towards infinity at both edges, or it goes downwards towards negative infinity at both edges. Since the leading coefficient is positive, the graph goes upwards at both edges. Thus, we begin by drawing just these pieces.

Next, we mark all the x-intercepts as points on the x-axis.

Ordinarily, since it is easy to compute, we would next determine the y-intercept by substituting $x = 0$ into the formula for the polynomial. However, in this example we already know that (0,0) is a point on the graph. That is, the y-intercept is also an x-intercept, or, the graph passes through the origin.

Starting from the left edge we connect to (-3,0), where we draw the graph "bouncing" back from the x-axis in a parabola like fashion.

Even the graph initially increases as x-increases just to the right of $x = -3$, we know that the graph returns to intersect the x-axis at $x = -1$, so guessing (your guess, as long as it is reasonable, will be just as good as anyone's) how high and where the peak occurs, we draw the graph as changing direction and heading down towards (-1,0).

Because the factor $(x + 1)$ has multiplicity 3, we draw the intersection at $x = -1$ just like that at $x = 0$ for the graph of $-x^3$. Thus, the graph flattens out as x draws near to $x = -1$ from the left, crosses at $x = -1$, and while initially flat to the right of $x = -1$, it then becomes steeper in descent.

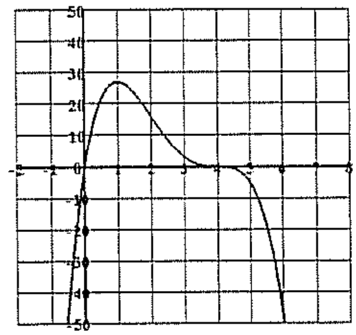
Somewhere between $x = -1$ and $x = 0$ the graph must change direction, *i.e.*, have a valley bottom, and then rise back to intersect the x-axis at $x = 0$. How deep and at what value of x you draw the bottom is not of concern at this time, we just know that it is there, not where it is precisely.

Because the factor x has multiplicity 1, the graph will cross through (0,0) like an oblique line. Again, somewhere between $x = 0$ and $x = 1$ the graph must change direction because it will intersect the x-axis at both of these value. This time the change in direction will be a peak, whose exact location we are unable to determine at this time.

Because the factor $(x - 1)$ has multiplicity 1, the graph will cross through (1,0) like an oblique line. Again, somewhere between $x = 1$ and $x = 2$, the graph will have to change direction, this time creating a valley bottom. Just as before, the shape of this bottom, either the x or y coordinates of such a point, cannot be determined at this time.

We draw the intersection at $x = 2$, like that of the curve x^3 at $x = 0$, and then connect up to our right edge piece.

2. Given the graph of a polynomial function g on the right, deduce a possible formula for g. Explain your reasoning, step by step.



Solutions:

First, we note that at both edges (left and right) the graph of the polynomial heads towards $-\infty$. Thus, we can deduce that the degree of the polynomial is even (since odd degree polynomials have graphs that at the edges head in opposite directions, only even degree polynomials head in the same direction at both edges. Also, the leading coefficient must be negative, otherwise at both edges the graph would head towards ∞, not $-\infty$.

Next, we note that the graph has two x-intercepts: $x = 0$ and $x = 4$. This means that the function contains the factors x and $(x - 4)$. Since the graph crosses through the intercept at $x = 0$ just like an oblique line, the multiplicity of the factor x is just 1. Since the graph crosses through the intercept at $x = 4$ like a cubic, the multiplicity of the factor $(x - 4)$ is 3 (or 5 or 7 etc., but 3 is simplest so our first choice). Thus, we can think of $g(x)$ as equal to $a\,x\,(x - 4)^3$, where a is the leading coefficient, which we already decided must be negative. Note that the degree of this polynomial will be 4, an even number, just as reasoned it must be above.

If we are to set a value for a, we must find one point on the graph other than the intercepts. The easiest to read would be vertices on the grid, but regrettably there do not seem to be any. There seem to be a few that we can approximate fairly accurately, say (½,20) or (2,15). Suppose we use the second one: (2,15). If we substitute this information into $g(x) = a\,x\,(x - 4)^3$ we obtain: $15 = a(2)(2 - 4)^3 = 2a(-2)^3 = -16a$. Looking at this, probably the y-value on the graph was 16, not 15, so that $16 = -16a \Rightarrow a = -1$.

What would the other point have yielded? We obtain:
$20 = a(½)(½ - 4)^3 = ½a(-7/2)^3 \Rightarrow 40 = a(-343/8) \Rightarrow a = -(320/343)$. While this is not exactly the same as before, it is similar. We note that the difference is due to our inability to read precise values off this grid, and so this is "as good as it gets".

3. Given the function $h(x) = -4x^5 - 20x^4 - 9x^3 + 106x^2 + 212x + 120$:
   a. Determine the x-intercepts of $h(x)$.
   b. **<u>Without</u>** computing a table of values for the function, **<u>sketch</u>** a graph of $h(x)$, explaining how it accurately portrays all information gleaned in a. and b. above.

Solutions:
Doing this will require factoring $h(x)$, but cleverness will spare us too much hard work. If $x = (p/q)$ is a rational root, with p and q having no factors in common, then p divides evenly into the constant term, 120, and q divides evenly into the leading coefficient, -4. Using this we can set up a list of all possible rational roots. To test if a particular number on the list is a root, we just substitute it into $h(x)$ and see if we obtain 0 (for a root), or not (if it is not a root).

$p = \pm\{1,2,3,4,5,6,8,10,12,15,20,24,30,40,60,120\}, q = \pm\{1,2,4\}$

$$\frac{p}{q} = \pm\frac{1}{1},\pm\frac{2}{1},\pm\frac{3}{1},\pm\frac{4}{1},\pm\frac{5}{1},\pm\frac{6}{1},\pm\frac{8}{1},\pm\frac{10}{1},\pm\frac{12}{1},\pm\frac{15}{1},\pm\frac{20}{1},\pm\frac{24}{1},\pm\frac{30}{1},\pm\frac{40}{1},\pm\frac{60}{1},\pm\frac{120}{1},$$
$$\pm\frac{1}{2},\pm\frac{2}{2},\pm\frac{3}{2},\pm\frac{4}{2},\pm\frac{5}{2},\pm\frac{6}{2},\pm\frac{8}{2},\pm\frac{10}{2},\pm\frac{12}{2},\pm\frac{15}{2},\pm\frac{20}{2},\pm\frac{24}{2},\pm\frac{30}{2},\pm\frac{40}{2},\pm\frac{60}{2},\pm\frac{120}{2},$$
$$\pm\frac{1}{4},\pm\frac{2}{4},\pm\frac{3}{4},\pm\frac{4}{4},\pm\frac{5}{4},\pm\frac{6}{4},\pm\frac{8}{4},\pm\frac{10}{4},\pm\frac{12}{4},\pm\frac{15}{4},\pm\frac{20}{4},\pm\frac{24}{4},\pm\frac{30}{4},\pm\frac{40}{4},\pm\frac{60}{4},\pm\frac{120}{4}$$

Clearly this list is daunting - who wants to go through all of these? The key is, or will be, once one root has been determined, to divide the corresponding factor out, and create a new list that is shorter.

$h(1) = -4 - 20 - 9 + 106 + 212 + 120 \neq 0$
$h(-1) = 4 - 20 + 9 + 106 - 212 + 120 \neq 0$
$h(2) = -128 - 320 - 72 + 424 + 424 + 120 \neq 0$
$h(-2) = 128 - 320 + 72 + 424 - 424 + 120 = 0$

Thus, $(x+2)$ is a factor, so we do the long division.

$$\begin{array}{r} -4x^4 - 12x^3 + 15x^2 + 76x + 60 \\ x+2 \overline{)\,-4x^5 - 20x^4 - 9x^3 + 106x^2 + 212x + 120} \\ \underline{-4x^5 - 8x^4} \qquad\qquad\qquad\qquad\qquad\quad \\ -12x^4 - 9x^3 \qquad\qquad\qquad\qquad \\ \underline{-12x^4 - 24x^3} \qquad\qquad\qquad\qquad \\ +15x^3 + 106x^2 \qquad\qquad\quad \\ \underline{+15x^3 + 30x^2} \qquad\qquad\quad \\ 76x^2 + 212x \qquad \\ \underline{76x^2 + 152x} \qquad \\ 60x + 120 \\ \underline{60x + 120} \\ 0 \end{array}$$

Let $h_2(x) = -4x^4 - 12x^3 + 15x^2 + 76x + 60$. Now we need to determine zeroes, and factors, of $h_2(x)$. Note that values of x that were not zeroes for $h(x)$ when we tested before, cannot be zeroes of $h_2(x)$, so we need not test them again. The list of possible rational zeroes is now:

$p = \pm\{1,2,3,4,5,6,8,10,12,15,20,24,30,40,60\}, q = \pm\{1,2,4\}$

$$\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{5}{1}, \pm\frac{6}{1}, \pm\frac{8}{1}, \pm\frac{10}{1}, \pm\frac{12}{1}, \pm\frac{15}{1}, \pm\frac{20}{1}, \pm\frac{24}{1}, \pm\frac{30}{1}, \pm\frac{40}{1}, \pm\frac{60}{1},$$
$$\pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{3}{2}, \pm\frac{4}{2}, \pm\frac{5}{2}, \pm\frac{6}{2}, \pm\frac{8}{2}, \pm\frac{10}{2}, \pm\frac{12}{2}, \pm\frac{15}{2}, \pm\frac{20}{2}, \pm\frac{24}{2}, \pm\frac{30}{2}, \pm\frac{40}{2}, \pm\frac{60}{2},$$
$$\pm\frac{1}{4}, \pm\frac{2}{4}, \pm\frac{3}{4}, \pm\frac{4}{4}, \pm\frac{5}{4}, \pm\frac{6}{4}, \pm\frac{8}{4}, \pm\frac{10}{4}, \pm\frac{12}{4}, \pm\frac{15}{4}, \pm\frac{20}{4}, \pm\frac{24}{4}, \pm\frac{30}{4}, \pm\frac{40}{4}, \pm\frac{60}{4}$$

$h_2(-2) = -4(-2)^4 - 12(-2)^3 + 15(-2)^2 + 76(-2) + 60 = -64 + 96 + 60 - 152 + 60 = 0$

Thus, $(x+2)$ is again a factor.

$$\begin{array}{r} -4x^3 - 4x^2 + 23x + 30 \\ x+2 \overline{)\,-4x^4 - 12x^3 + 15x^2 + 76x + 60} \\ \underline{-4x^4 - 8x^3} \qquad\qquad\qquad\qquad \\ -4x^3 + 15x^2 \qquad\qquad\quad \\ \underline{-4x^3 - 8x^2} \qquad\qquad\quad \\ +23x^2 + 76x \qquad \\ \underline{+23x^2 + 46x} \qquad \\ 30x + 60 \\ \underline{30x + 60} \\ 0 \end{array}$$

Let $h_3(x) = -4x^3 - 4x^2 + 23x + 30$. Now we need to determine zeroes, and factors, of $h_3(x)$. Note that values of x that were not zeroes for $h(x)$ or $h_2(x)$ when we tested before, cannot be zeroes of $h_3(x)$, so we need not test them again. The list of possible rational zeroes is now:

$p = \pm\{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30\}, q = \pm\{1, 2, 4\}$

$$\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{5}{1}, \pm\frac{6}{1}, \pm\frac{8}{1}, \pm\frac{10}{1}, \pm\frac{12}{1}, \pm\frac{15}{1}, \pm\frac{20}{1}, \pm\frac{24}{1}, \pm\frac{30}{1},$$
$$\pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{3}{2}, \pm\frac{4}{2}, \pm\frac{5}{2}, \pm\frac{6}{2}, \pm\frac{8}{2}, \pm\frac{10}{2}, \pm\frac{12}{2}, \pm\frac{15}{2}, \pm\frac{20}{2}, \pm\frac{24}{2}, \pm\frac{30}{2},$$
$$\pm\frac{1}{4}, \pm\frac{2}{4}, \pm\frac{3}{4}, \pm\frac{4}{4}, \pm\frac{5}{4}, \pm\frac{6}{4}, \pm\frac{8}{4}, \pm\frac{10}{4}, \pm\frac{12}{4}, \pm\frac{15}{4}, \pm\frac{20}{4}, \pm\frac{24}{4}, \pm\frac{30}{4}$$

$h_3(-2) = -4(-2)^3 - 4(-2)^2 + 23(-2) + 30 = 32 - 16 - 46 + 30 = 0$. Thus, $(x + 2)$ is again a factor.

$$\begin{array}{r} -4x^2 + 4x + 15 \\ x+2\overline{\big)\,-4x^3 - 4x^2 + 23x + 30} \\ \underline{-4x^3 - 8x^2} \qquad\qquad \\ 4x^2 + 23x \qquad \\ \underline{4x^2 + \ \ 8x} \qquad \\ +15x + 30 \\ \underline{+15x + 30} \\ 0 \end{array}$$

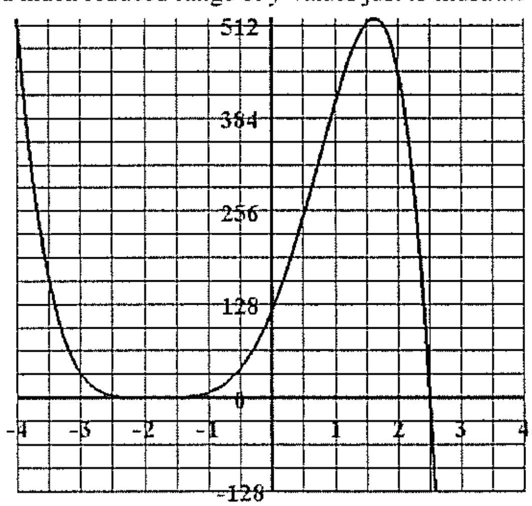
Let $h_4(x) = -4x^2 + 4x + 15 = -(4x^2 - 4x - 15) = -(2x + 3)(2x - 5)$

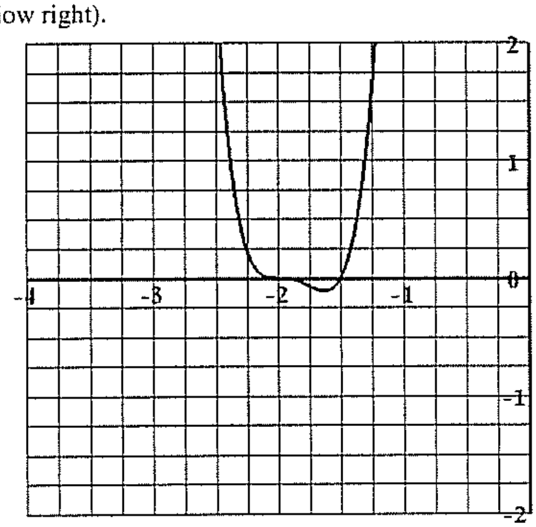
Thus, $h(x) = -4x^5 - 20x^4 - 9x^3 + 106x^2 + 212x + 120 = -(x + 2)^3(2x + 3)(2x - 5)$

This means that the function had three x-intercepts: $x = -2$ (with multiplicity 3), -3/2 (multiplicity 1), and 5/2 (multiplicity 1). The multiplicity of each factor tells us how the graph of the function intersects the x-axis, like a cubic in the case of $x = -2$, and like an oblique line in the other two cases.

Since the degree of $h(x)$, 5, is odd, at one edge the function heads towards $-\infty$, at the other it heads towards ∞. Since the leading coefficient of $h(x)$, -4, is negative, at the left edge the graph of $h(x)$ heads towards ∞ at the left edge, and towards $-\infty$ at the right edge.

In the first graph (below left), to show the overall shape we have to have a large range of y-values. Unfortunately, this obscures what the graph looks like near the x-intercepts at $x = -2$ and -3/2. Thus, we draw a separate graph with a much reduced range of y-values just to illustrate that region (below right).





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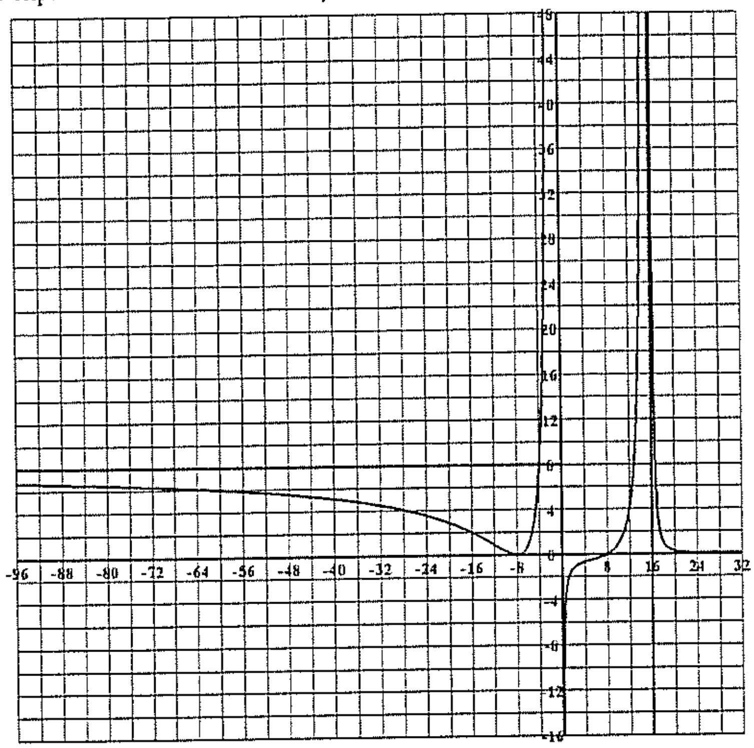
In the problem below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of this sheet.

Below right you are given a graph of a function, $f(x)$.

a. Explain, in sentences in English, each step of the process that you use to deduce parts of a graph of the reciprocal of $f(x)$. Using a numbering system to label each step.

b. Draw a graph of the reciprocal of the function whose graph is given. Label each part of the graph with the numbers that you used in a. above to explain the derivation of that part.

Solutions:



In the problem below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of this sheet.

Below right you are given a graph of a function, $f(x)$.

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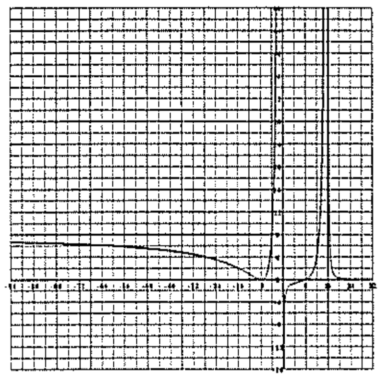
b. Draw a graph of the reciprocal of the function whose graph is given. Label each part of the graph with the numbers that you used in a. above to explain the derivation of that part.



Solutions:

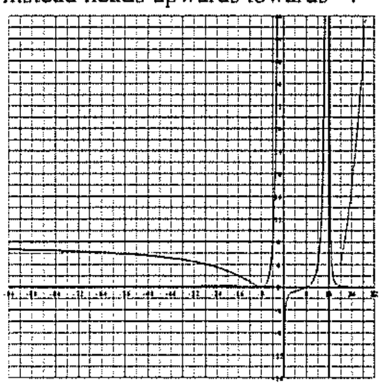
a. The steps, in order of use, are (pertaining to the given graph of $f(x)$):

1. examine edge behaviour;
2. locate x-intercepts;
3. note peaks and valleys;
4. locate vertical asymptotes;
5. locate points of intersection of $f(x)$ and its reciprocal;
6. connect the pieces.

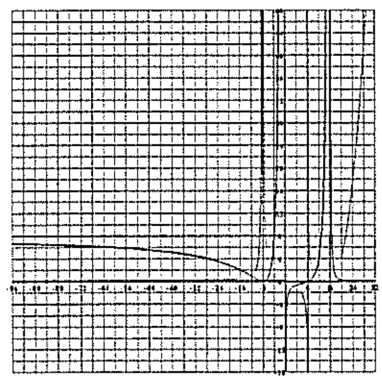
Now we carry these steps out.

1. At the left edge we note that the given graph of $f(x)$ has the line $y = 8$ as a horizontal asymptote. The given graph rises up away from the x-axis to get closer and closer to $y = 8$ at this left edge. Based on this information, we deduce that the reciprocal function has a horizontal asymptote at the left edge, $y = (1/8)$, and that the graph will come down towards the line (and hence the x-axis), as x values increase in size (absolute value) at the left edge.

At the right edge we note that the given graph of $f(x)$ is asymptotic to the x-axis, *i.e.*, $y = 0$. Based on this information, we deduce that the reciprocal function does not have a horizontal asymptote at the right edge, but instead heads upwards towards ∞.



1. edges

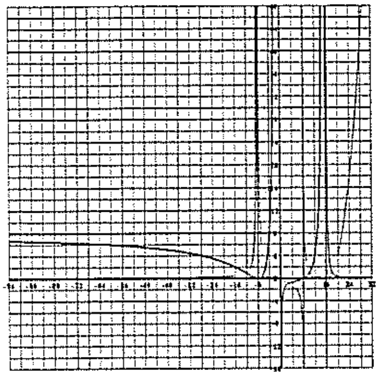


2. x-intercepts → V.A.

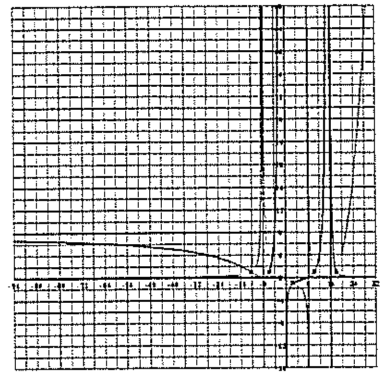
2. The graph of $f(x)$ has two x-intercepts: $x = -8$ and 8. Each of these intercepts corresponds to a vertical asymptote in a graph of the reciprocal function. Note that nearby $x = -8$ the given graph of $f(x)$ is above the x-axis, hence our graph of the reciprocal function will also be above the x-axis.This means that our graph of the reciprocal function should head upwards towards ∞ on both sides of the vertical line, $x = -8$. To the left of but nearby $x = 8$ we note that the given graph of $f(x)$ is below the x-axis, hence our graph of the reciprocal function will also be below the x-axis nearby and to the left of $x = 8$, so this graph will head down towards $-\infty$. To the right of but nearby $x = 8$ we note that the given graph of $f(x)$ is above the x-axis, hence our graph of the reciprocal function will also be above the x-axis nearby and to the right of $x = 8$, so this graph will head up towards ∞.

3. The given graph has no peaks, and the only valley is also an x-intercept, $x = -8$, and so as analysed above, the reciprocal function has a vertical asymptote at that value of x.

4. The given graph has two vertical asymptotes, at $x = 0$ and $x = 16$. As x gets close to each of these values, $f(x)$ gets close to $\pm\infty$, and the given graph is heading away from the x-axis. Thus, a graph of the reciprocal function will approach the x-axis, getting closer and closer to 0, while $|f(x)|$ gets larger and larger. In fact, the reciprocal function will have missing points, missing x-intercepts, (0,0) and (16,0). For the first of these two, (0,0), the graph will cut through the x-axis, missing the actual point of intersection, however, at the second point, (16,0), the graph will "bounce" from the x-axis, like an even powered power function, again missing the actual point of intersection.



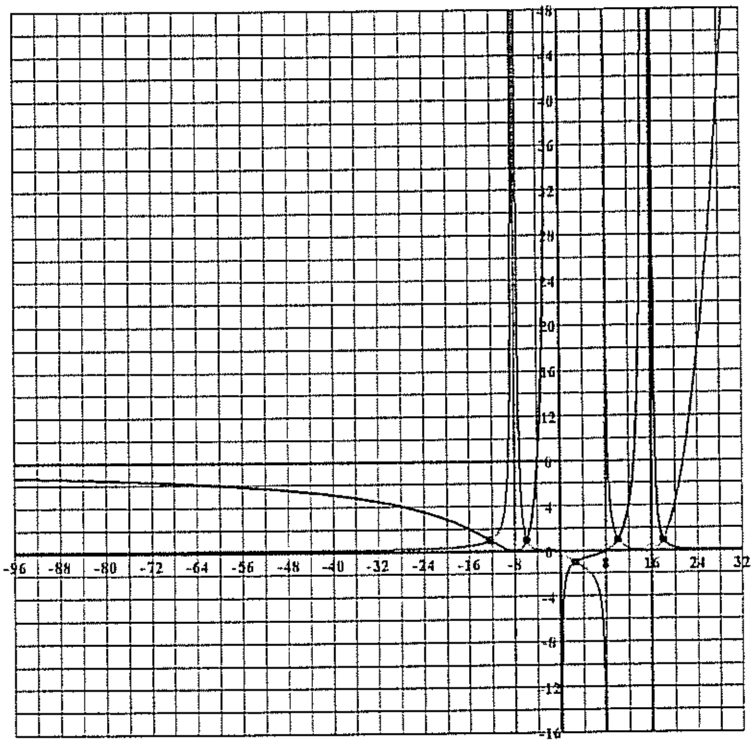
4. V.A. ⇒ M.P.(x-intercept)



5. points of intersection

5. We imagine horizontal lines $y = -1$ and $y = 1$, and see that such lines intersect the given graph approximately at (-12,1), (-6,1), (2,-1), (10,1), and (18,1). Since $1/1 = 1$ and $1/(-1) = -1$, these five points are the points where a graph of the reciprocal function should intersect the given graph of $f(x)$.

6. We draw the graph connecting the pieces.



6. put it all together

Worksheet on Rational Functions

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In the problem below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of this sheet.

You are given the function, $f(x)=\dfrac{(x+3)(2x+1)^2(x-2)^3}{(x+3)^2x(x-2)^3}$

a. Determine the function $h(x)$ which is the ratio of leading terms of the numerator and denominator of $f(x)$. Reduce it algebraically. Explain how the graph of $f(x)$ will behave at the edges, and how you reasoned this out.

b. Determine all zeroes of $p(x)$, the numerator of $f(x)$. Determine all zeroes of $q(x)$, the denominator of $f(x)$.

c. Determine the function $g(x)$ by simplifying $f(x)$ algebraically.

d. Determine all x-intercepts, all missing points, and all vertical asymptotes of $f(x)$. Show your work step by step and explaining as you go.

e. Draw a graph of $f(x)$ that illustrates all the information that you have deduced in a. - d. above.

Solutions:

In the problem below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of this sheet.

You are given the function, $f(x)=\frac{(x+3)(2x+1)^2(x-2)^3}{(x+3)^2x(x-2)^3}$

a. Determine the function $h(x)$ which is the ratio of leading terms of the numerator and denominator of $f(x)$. Reduce it algebraically. Explain how the graph of $f(x)$ will behave at the edges, and how you reasoned this out.
b. Determine all zeroes of $p(x)$, the numerator of $f(x)$. Determine all zeroes of $q(x)$, the denominator of $f(x)$.
c. Determine the function $g(x)$ by simplifying $f(x)$ algebraically.
d. Determine all x-intercepts, all missing points, and all vertical asymptotes of $f(x)$. Show your work step by step and explaining as you go.
e. Draw a graph of $f(x)$ that illustrates all the information that you have deduced in a. - d. above.

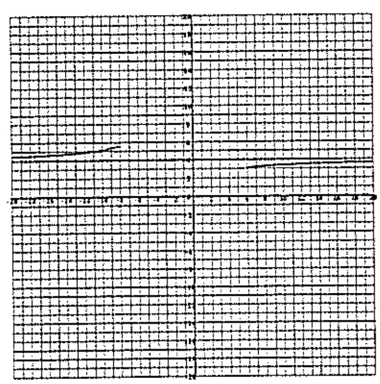
Solutions:

a. $h(x)=\frac{x(2x)^2(x)^3}{(x)^2x(x)^3}=\frac{4\cancel{x^6}}{\cancel{x^6}}=4$

We know that for a polynomial function, at the edges the polynomial function and the leading term have similar values, hence the shape of the graph of the polynomial function at the edges is the same as that of the leading term. For a rational function, the ratio of the leading terms of the numerator and the denominator helps us to predict the behaviour of the rational function at the edges. In this example, since that ratio is a constant, 4, we interpret this as meaning that at the edges a graph of $f(x)$ will look more and more like the constant function, $y = 4$, *i.e.*, the function $f(x)$ has the line $y = 4$ as horizontal asymptote at both edges.
To determine, independently at each edge, whether a graph of $f(x)$ will approach the horizontal asymptote, $y = 4$, from above or from below we can use either of two tactics: a numerical approach; an algebraic approach.
The numeric approach consists of substituting in a sequence of large values of x, to see if values of the function $f(x)$ are larger than (graph is above asymptote) or smaller than (graph is below asymptote) 4. Suppose we try this for the left edge. The small table below shows what we have determined.

x	-1,000,000		-10,000		-100
$f(x)$	4.000008000		4.000800250		4.082577320

From the table value it seems clear that the graph we draw should approach $y = 4$ from above at the left edge.
The algebraic approach consists of looking at the ratio of not just leading terms, but the two highest power terms in both numerator and denominator. In this case we have $h^*(x)=\frac{4x^2+4x}{x^2+3x}=\frac{4\cancel{x}(x+1)}{\cancel{x}(x+3)}=4\left(\frac{x+1}{x+3}\right)$. For large positive values of x we can see that the denominator will be larger than the numerator, hence the fraction in brackets is less than 1, hence the overall value of h^* will be less than (graph is below asymptote) 4 at the right edge of the graph.



b. The numerator of $f(x)$, $(x + 3)(2x + 1)^2(x - 2)^3$, is zero at $x = -3, -½, 2$.
The denominator of $f(x)$, $(x + 3)^2x(x - 2)^3$, is zero at $x = -3, 0, 2$.

c. The function $g(x)$, obtained by cancelling common factors from the numerator and denominator of $f(x)$, is $g(x)=\frac{(2x+1)^2}{(x+3)x}$.

d. The zeroes of the numerator that are not zeroes of the denominator are the x-intercepts of $f(x)$. There is only one such zero, $x = -½$, and since the factor corresponding to it, $(2x + 1)$, has multiplicity 2, an even number, the graph should "bounce" back from the x-axis, and the function will not change sign at this x-intercept.

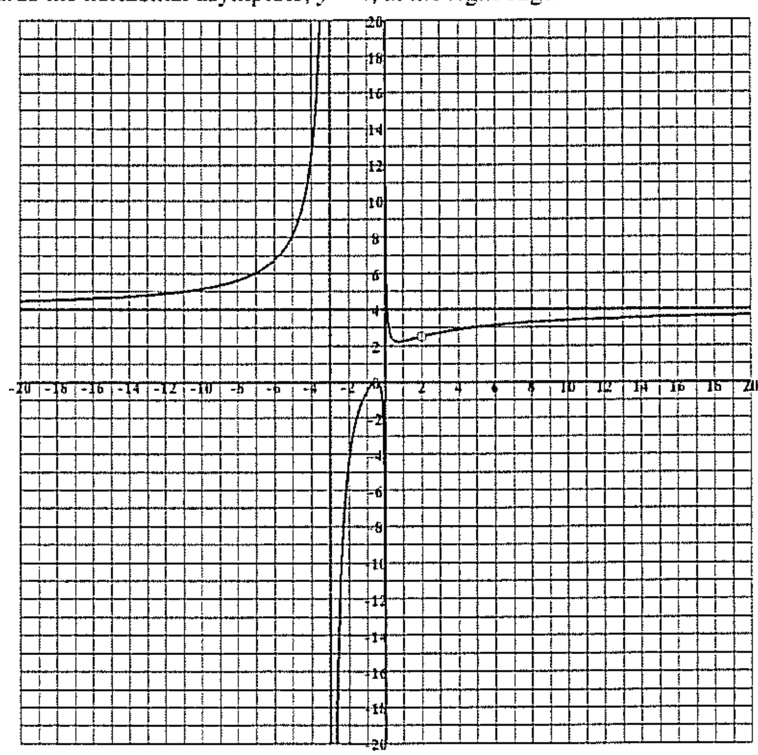
The zeroes of the denominator that are not zeroes of the numerator are vertical asymptotes of $f(x)$. There is one such zero, $x = 0$, and since the factor corresponding to it, x, has multiplicity 1, an odd number, the function will change sign across the vertical asymptote, *i.e.*, on one side of $x = 0$ the graph of $f(x)$ will head towards $-\infty$, and on the other it will head towards ∞.

The other two zeroes of the numerator and denominator, $x = -3$ and 2, are common, so we must examine $g(x)$ to determine the behaviour of $f(x)$ around these values of x. We note that $x = -3$ is a zero of the denominator only of $g(x)$, hence $g(x)$ has a vertical asymptote at $x = -3$. But $g(x)$ and $f(x)$ can differ only at the single points $x = -3$ and 2, thus where $g(x)$ has a vertical asymptote, so will $f(x)$. Since the factor causing the vertical asymptote, $(x + 3)$, has multiplicity 2, an even number, the graphs of $g(x)$ and $f(x)$ will not change sign across the vertical asymptote, *i.e.*, on both sides of $x = -3$ the graphs will head down towards $-\infty$ or they will head up towards ∞. We note that $g(2) = 5/2$. Since $f(x)$ was undefined at $x = 2$, we interpret this as meaning that the graph of $f(x)$ will have a missing point, (2,5/2).

e. Having placed all information in a table as it was determined, we now put it all together to draw a graph of $f(x)$:

	H.A.			V.A. mult. 2		x-int. mult. 2			V.A. mult. 1			M.P.		H.A.
x	$-\infty$		-3^-	-3	-3^+	-½		0^-	0	0^+		2		∞
$f(x)$	4		$-\infty$	U	$-\infty$	0		$-\infty$		∞		2.5		4
	□ ‾‾		↘	|	↙	⌢		↘	|	↖		○		‾‾ □

1. Since at the left edge the graph is above the y-axis, and on the interval $(-\infty,-3)$ there are no x-intercepts, as the graph approaches -3 from the left, the graph must be heading upwards towards ∞, not downwards towards $-\infty$ because to do that the graph would have to cross the x-axis somewhere on $(-\infty,-3)$.
2. Since there is an even multiplicity for the factor in the denominator that causes the vertical asymptote at $x = -3$, so the graph does not change sign across the vertical line $x = -3$. Thus, on the right of $x = -3$, the graph must head downwards towards $-\infty$.
3. Since the graph is below the x-axis immediately to the right of $x = -3$, and there are no x-intercepts on the interval (-3,-½), the graph will approach the x-intercept at $x = -½$ from below. Since there is an even multiplicity for the factor in the numerator that causes the x-intercept at $x = -½$, the graph will bounce back and stay below the x-axis.
4. Since the graph is below the x-axis immediately to the right of $x = -½$, and there are no x-intercepts on the interval (-½,0), the graph will approach the vertical asymptote $x = 0$ heading downwards towards $-\infty$.
5. Since there is an odd multiplicity for the factor in the denominator that causes the vertical asymptote at $x = 0$, so the graph changes sign across the vertical line $x = 0$. Thus, on the right of $x = 0$, the graph must head upwards towards ∞.
6. Since there are no x-intercepts on the interval $(0,\infty)$, the graph will not cross or even touch the x-axis on that interval. It will pass through the missing point (2,2.5) and somewhere on the interval reach the bottom of a valley, and then head up towards the horizontal asymptote, $y = 4$, at the right edge.



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Name: ______________________ Name: ______________________

In the problems below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of this sheet. This worksheet is to be done cooperatively, working in groups of four.

1. Given the function $f(x)=\dfrac{\sqrt{2x+1}}{3}$:

   a. Break this function into a sequence of single operation steps, starting with x on the left and ending with $\dfrac{\sqrt{2x+1}}{3}$ on the right, as done in class.

   ____

   b. Write out the reverse sequence of single operation steps, starting with $\dfrac{\sqrt{2x+1}}{3}$ on the right and ending with x on the left, as done in class.

   ____

   c. Write out the reverse sequence of single operation steps, starting with x on the right and ending with whatever on the left, as done in class.

   ____

   d. Write out the formula for $f^{-1}(x)$.

   ____

2. On the right is the graph of a rational function, $f(x)$.

   a. Your first task is to explain, using complete sentences, how to use that graph to compute values of $f^{-1}(x)$.

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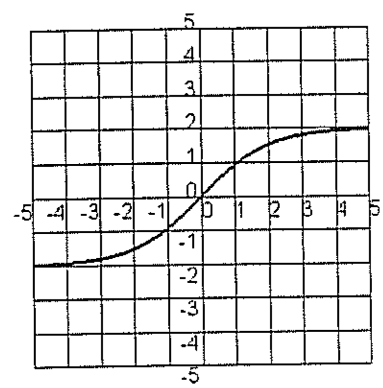


   b. Your second task is to explain, using complete sentences, why we do not generally refer to this graph of $f(x)$ as the graph of $f^{-1}(x)$.

   ____

   ____

   ____

   c. Your third task is to draw the usual graph of $f^{-1}(x)$ on the same set of axes that we see the graph of $f(x)$ on. On the lines below write one complete sentence explaining how you do this.

   ____

   ____

   ____

In the problems below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of this sheet. This worksheet is to be done cooperatively, working in groups of four.

1. Given the function $f(x)=\dfrac{\sqrt{2x+1}}{3}$:

   a. Break this function into a sequence of single operation steps, starting with x on the left and ending with $\dfrac{\sqrt{2x+1}}{3}$ on the right, as done in class.

Solution:

$$x \xrightarrow{\times 2} 2x \xrightarrow{+1} 2x+1 \xrightarrow{\sqrt{(\ )}} \sqrt{2x+1} \xrightarrow{\div 3} \frac{\sqrt{2x+1}}{3}$$

   b. Write out the reverse sequence of single operation steps, starting with $\dfrac{\sqrt{2x+1}}{3}$ on the right and ending with x on the left, as done in class.

Solution:

$$x \xleftarrow{\div 2} 2x \xleftarrow{-1} 2x+1 \xleftarrow{(\ )^2} \sqrt{2x+1} \xleftarrow{\times 3} \frac{\sqrt{2x+1}}{3}$$

   c. Write out the reverse sequence of single operation steps, starting with x on the right and ending with whatever on the left, as done in class.

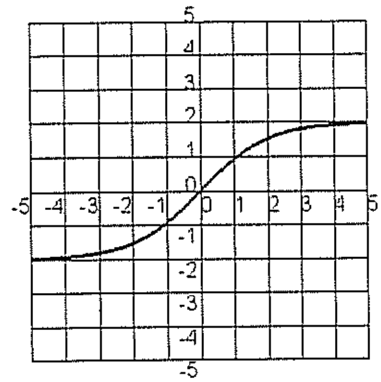
Solution:

$$f^{-1}(x)=\frac{9x^2-1}{2} \xleftarrow{\div 2} 9x^2-1 \xleftarrow{-1} 9x^2 \xleftarrow{(\ )^2} 3x \xleftarrow{\times 3} x$$

   d. Write out the formula for $f^{-1}(x)$.

Solution:

$$f^{-1}(x)=\frac{9x^2-1}{2}$$



2. On the right is the graph of a rational function, $f(x)$.
   a. Your first task is to explain, using complete sentences, how to use that graph to compute values of $f^{-1}(x)$.

Solution:

Given the graph of $f(x)$ we use it to compute values of $f(x)$ as follows: given a value of x, say 2, we draw a vertical line ($x = 2$), and then locate the point of intersection of that line with the graph; next, we draw a horizontal line through the point of intersection and locate the intersection of the horizontal line with the y-axis; the y-coordinate on the y-axis of this latter point of intersection is $f(2)$, which on the adjacent graph appears to be about 1.5. We can determine values of $f^{-1}(x)$ by reversing this procedure. That is: given a value of x, say 1.5, we draw a horizontal line ($y = 1.5$) and then locate the point of intersection of that line with the graph; next, we draw a vertical through the point of intersection and locate the intersection of the vertical line with the x-axis; the x-coordinate on the x-axis of this latter point of intersection is $f^{-1}(1.5)$, which on the adjacent graph appears to be 2. In a nutshell, by thinking of the graph as having independent variable on the vertical axis (domain values) and dependent variable on the horizontal axis (range values), we use the general procedure for determining values of a function from its graph.

   b. Your second task is to explain, using complete sentences, why we do not generally refer to this graph of $f(x)$ as the graph of $f^{-1}(x)$.

Solution:

From the discussion above we see that the graph of $f(x)$ can be used as the graph of $f^{-1}(x)$, provided we understand that the domain of $f^{-1}(x)$ lies on the vertical axis and its range lies on the horizontal axis, a complete reversal of standard practice. Thus, if we made this mental adjustment we would have no need to draw any additional graph for $f^{-1}(x)$. The problem is that making that mental adjustment is not so simple, especially if we want to compare the graph of $f(x)$ to that of $f^{-1}(x)$. Thus, we draw a new graph to represent $f^{-1}(x)$ to avoid the confusion associated with an independent variable being vertical while the dependent variable is horizontal.

   c. Your third task is to draw the usual graph of $f^{-1}(x)$ on the same set of axes that we see the graph of $f(x)$ on. On the lines below write one complete sentence explaining how you do this.

Solution:

Since the graph of $f(x)$ can actually serve as the graph of $f^{-1}(x)$, with the roles of the axes reversed, if we want to see what the graph of $f^{-1}(x)$ looks like with the roles of the axes not reversed, we need to take the graph of $f(x)$ and transform it in such a way that the roles of the axes are reversed, *i.e.*, so that the positive y-axis lies upon the positive x-axis, and the positive x-axis lies upon the positive y-axis. We notice that if we draw the graph of the line $y = x$, the so called "45° line", and use this line as a mirror to reflect everything, then we obtain exactly the transformation that we wish, namely we swap y with x and vice versa. Thus, reflecting the graph of $f(x)$ through the line $y = x$ creates the graph of $f^{-1}(x)$, with domain now on the horizontal axis and range on the vertical axis, *i.e.*, in the standard way.

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This worksheet is to be done cooperatively, working in groups of four.

1. Given the function $f_0(x) = 2^x$, and the fact that $f_5(x) = -(\frac{1}{2})2^{\frac{1}{2}(x+1)} - 1$
   a. Write a sequence of functions, $f_i(x)$, $i = 1, 2, 3,$ and 4, where from one function to the next is a **SINGLE STEP** (shift, stretch/compress, or flip).

Solution:

$f_1(x) =$

$f_2(x) =$

$f_3(x) =$

$f_4(x) =$

   b. For each of $f_1(x), f_2(x), f_3(x), f_4(x)$ and $f_5(x)$, explain in words what transformation of the graph of the previous function is required to draw a graph of this new function.

Solution:

$f_1(x)$ from $f_0(x)$: ______________________

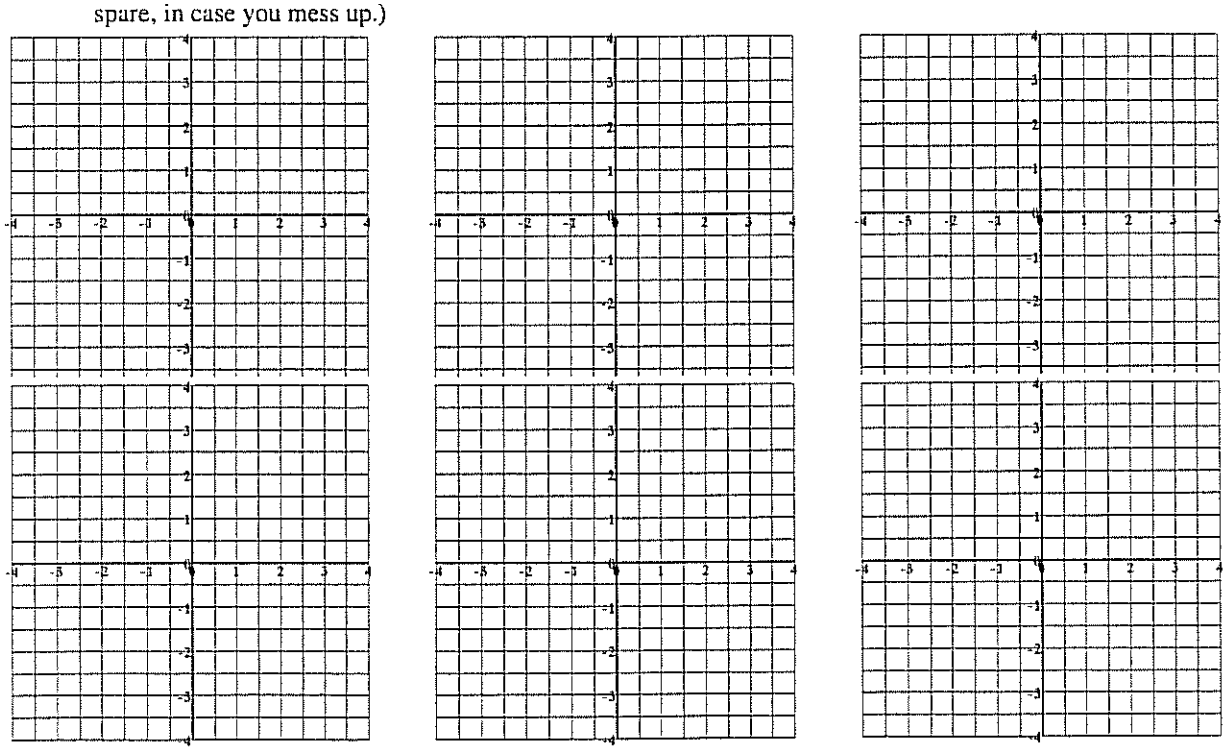
$f_2(x)$ from $f_1(x)$: ______________________

$f_3(x)$ from $f_2(x)$: ______________________

$f_4(x)$ from $f_3(x)$: ______________________

$f_5(x)$ from $f_4(x)$: ______________________

   c. Below there are six grids that you will use to illustrate the graphs of $f_0(x)$ to $f_5(x)$. On your first grid draw a **pair** of graphs, $f_0(x)$ and $f_1(x)$, using different colours and clearly labelling the two graphs. When plotting $f_0(x)$ show accurately the points $(-1,f_0(-1))$, $(0,f_0(0))$ and $(1,f_0(1))$, and then just sketch the shape of $f_0(x)$. When plotting $f_1(x)$ show accurately the three points corresponding to the above three, and then just sketch the shape of $f_1(x)$. On the next grid, copy your graph of $f_1(x)$ and then repeat the procedure above to generate a sketch of $f_2(x)$. On the next grid, copy your graph of $f_2(x)$, and then repeat the procedure above to generate a sketch of $f_3(x)$. Continue this process until the fifth grid shows $f_4(x)$ and $f_5(x)$. (One grid is a spare, in case you mess up.)



2. Given the piecewise defined function, $f(x)=\begin{cases} ax^2+bx+2 , & x<-1 \\ bx-a , & -1\leq x\leq 1 \\ 2ax+b+3 , & 1<x \end{cases}$, determine all possible values of a and b such that this function has no vertical jumps or missing points. Be sure to show your work and explain in words the logic that you are using.

Solution:

1. Given the function $f_0(x) = 2^x$, and the fact that $f_5(x) = -(½)2^{½(x+1)} - 1$
   a. Write a sequence of functions, $f_i(x)$, $i = 1, 2, 3$, and 4, where from one function to the next is a **<u>SINGLE STEP</u>** (shift, stretch/compress, or flip).

Solution:

$f_1(x) = f_0(x+1) = 2^{(x+1)}$

$f_2(x) = f_1(½x) = 2^{½(x+1)}$

$f_3(x) = (½)f_2(x) = (½)2^{½(x+1)}$

$f_4(x) = -f_3(x) = -(½)2^{½(x+1)}$

   b. For each of $f_1(x), f_2(x), f_3(x), f_4(x)$ and $f_5(x)$, explain in words what transformation of the graph of the previous function is required to draw a graph of this new function.

Solution:

$f_1(x)$ from $f_0(x)$: adding 1 to x causes the graph to shift horizontally one unit to the left

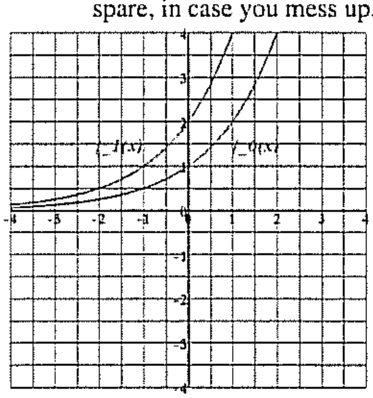
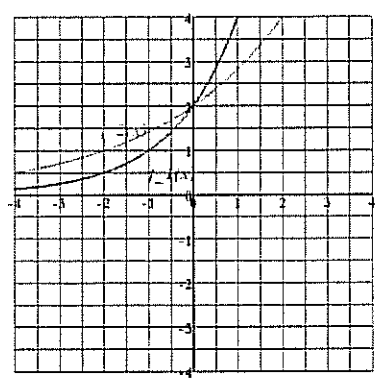
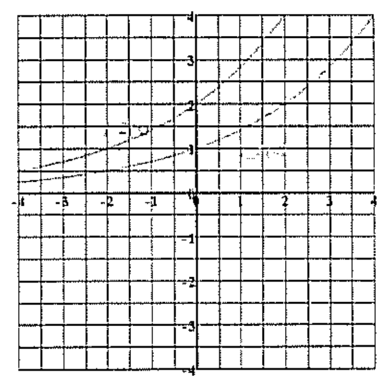
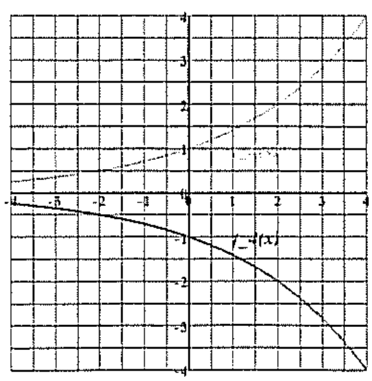
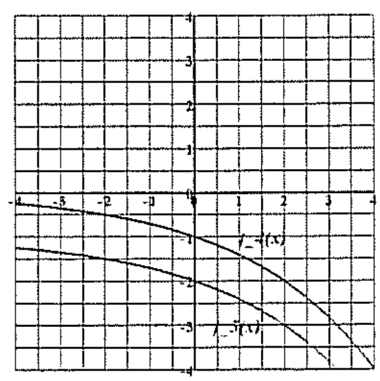
$f_2(x)$ from $f_1(x)$: multiplying x by ½ causes the graph to stretch horizontally away from $x = -1$ by a factor of 2

$f_3(x)$ from $f_2(x)$: multiplying y by ½ causes the graph to compress vertically towards $y = 0$ (the x-axis) by a factor of 2

$f_4(x)$ from $f_3(x)$: multiplying y by -1 causes the graph to flip (reflect) vertically about the line $y = 0$ (the x-axis)

$f_5(x)$ from $f_4(x)$: adding -1 to y causes the graph to shift vertically one unit downwards

   c. Below there are six grids that you will use to illustrate the graphs of $f_0(x)$ to $f_5(x)$. On your first grid draw a **<u>pair</u>** of graphs, $f_0(x)$ and $f_1(x)$, using different colours and clearly labelling the two graphs. When plotting $f_0(x)$ show accurately the points $(-1,f_0(-1))$, $(0,f_0(0))$ and $(1,f_0(1))$, and then just sketch the shape of $f_0(x)$. When plotting $f_1(x)$ show accurately the three points corresponding to the above three, and then just sketch the shape of $f_1(x)$. On the next grid, copy your graph of $f_1(x)$ and then repeat the procedure above to generate a sketch of $f_2(x)$. On the next grid, copy your graph of $f_2(x)$, and then repeat the procedure above to generate a sketch of $f_3(x)$. Continue this process until the fifth grid shows $f_4(x)$ and $f_5(x)$. (One grid is a spare, in case you mess up.)

2. Given the piecewise defined function, $f(x)=\begin{cases} ax^2+bx+2 & , \quad x<-1 \\ bx-a & , \quad -1\le x\le 1 \\ 2ax+b+3 & , \quad 1<x \end{cases}$, determine all possible values of a and b such that this function has no vertical jumps or missing points. Be sure to show your work and explain in words the logic that you are using.

Solution:

We notice that the given function is piecewise polynomial (the first piece, $f_1(x) = ax^2 + bx + 2$ is a quadratic, the second piece, $f_2(x) = bx - a$ is linear, and the third piece, $f_3(x) = 2ax + b + 3$ is also linear), hence within each piece there are no vertical jumps nor missing points. The only values of x at which such "disturbances" can happen are the "seams" of this graph, namely, $x = -1$ and $x = 1$.

We note that the function is actually defined at both of these seams, *i.e.*, $f(-1) = f_2(-1) = b(-1) - a = -b - a$ and $f(1) = f_2(1) = b(1) - a = b - a$, so there are no missing points.

To check if there is a vertical jump at $x = -1$, we need that $f_1(-1) = f_2(-1)$. That is, $a(-1)^2 + b(-1) + 2 = b(-1) - a$ or $a - b + 2 = -b - a$ or $2a = -2$ or $a = -1$.

To check if there is a vertical jump at $x = 1$, we need that $f_2(1) = f_3(1)$. That is, $b(1) - a = 2a(1) + b + 3$ or $b - a = 2a + b + 3$ or $-3a = 3$ or $a = -1$.

Thus, we note that as long as $a = -1$, the function $f(x)$ will not have any missing points nor any vertical jumps. The value of b can be anything since it is eliminated from the equations.

Name: ______________________ Name: ______________________

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1. Given the polynomial function $f_0(x) = (x+3)(x+1)(x-1)^2(x-3)$:
   a. suppose $f_1(x) = |(x+3)(x+1)(x-1)^2(x-3)|$
      i. using a colour other than black, sketch those parts of a graph of $f_1(x)$ that differ from that of $f_0(x)$, using one of the grids below that already shows a graph of $f_0(x)$ in black;
      ii. in the space below write a definition of $f_1(x)$ as a piecewise defined function;
      iii. explain in simple sentences how you determined what the graph of $f_1(x)$ should look like, and then how you used this to give the definition of $f_1(x)$ as a piecewise defined function.

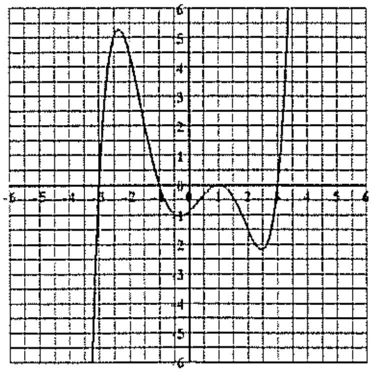
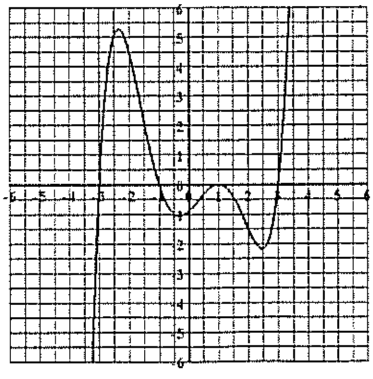
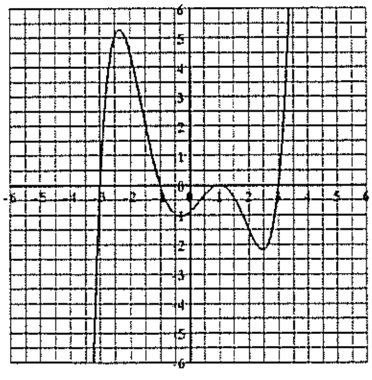
Solution:

   b. suppose $f_2(x) = (x+3)(x+1)|(x-1)^2|(x-3)$
      i. using a colour other than black, sketch those parts of a graph of $f_2(x)$ that differ from that of $f_0(x)$, using one of the grids below that already shows a graph of $f_0(x)$ in black;
      ii. in the space below write a definition of $f_2(x)$ as a piecewise defined function;
      iii. explain in simple sentences how you determined what the graph of $f_2(x)$ should look like, and then how you used this to give the definition of $f_2(x)$ as a piecewise defined function.

Solution:

   c. suppose $f_3(x) = |(x+3)|(x+1)(x-1)^2|(x-3)|$
      i. using a colour other than black, sketch those parts of a graph of $f_3(x)$ that differ from that of $f_0(x)$, using one of the grids below that already shows a graph of $f_0(x)$ in black;
      ii. in the space below write a definition of $f_3(x)$ as a piecewise defined function;
      iii. explain in simple sentences how you determined what the graph of $f_3(x)$ should look like, and then how you used this to give the definition of $f_3(x)$ as a piecewise defined function.

Solution:

2. Given the graph below of the function $f(x)$ determine all requested limits. No explanation is required, but bonus marks can be earned with good explanations.

a. $\lim_{x\to-\infty} f(x) =$

b. $\lim_{x\to-4^-} f(x) =$

c. $\lim_{x\to-4^+} f(x) =$

d. $\lim_{x\to-4} f(x) =$

e. $f(-4) =$

f. $\lim_{x\to0^-} f(x) =$

g. $\lim_{x\to0^+} f(x) =$

h. $\lim_{x\to0} f(x) =$

i. $f(0) =$

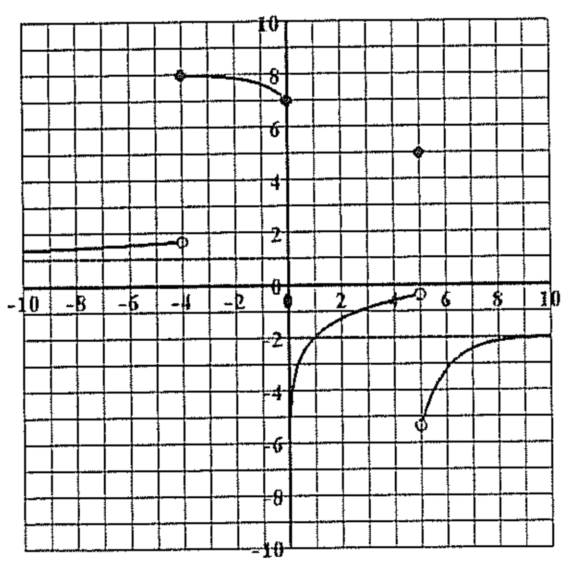
j. $\lim_{x\to5^-} f(x) =$

k. $\lim_{x\to5^+} f(x) =$

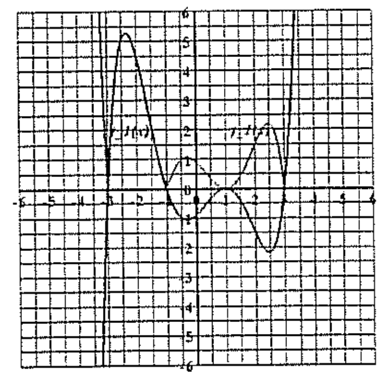
l. $\lim_{x\to5} f(x) =$

m. $f(5) =$

n. $\lim_{x\to\infty} f(x) =$



1. Given the polynomial function $f_0(x) = (x+3)(x+1)(x-1)^2(x-3)$:
   a. suppose $f_1(x) = |(x+3)(x+1)(x-1)^2(x-3)|$
      i. using a colour other than black, sketch those parts of a graph of $f_1(x)$ that differ from that of $f_0(x)$, using one of the grids below that already shows a graph of $f_0(x)$ in black;
      ii. in the space below write a definition of $f_1(x)$ as a piecewise defined function;
      iii. explain in simple sentences how you determined what the graph of $f_1(x)$ should look like, and then how you used this to give the definition of $f_1(x)$ as a piecewise defined function.

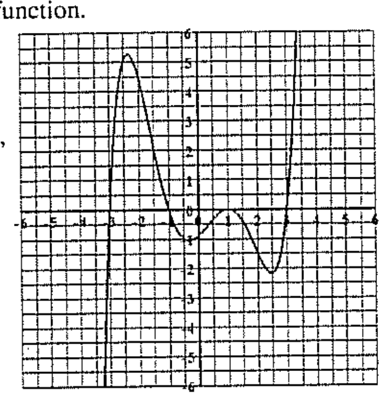


Solution:

iii. Since $f_1(x)$ is just the absolute value of $f_0(x)$, we know that wherever $f_0(x)$ is negative (graph is underneath the x-axis), that the graph of $f_1(x)$ will just be the reflection of the graph of $f_0(x)$ in the mirror of the x-axis. We also use this same idea to create the piecewise definition of $f_1(x)$.

ii. $$f_1(x) = \begin{cases} -f_0(x) & , \quad x < -3 \\ f_0(x) & , \quad -3 \le x \le -1 \\ -f_0(x) & , \quad -1 < x < 3 \\ f_0(x) & , \quad 3 \le x \end{cases}$$

   b. suppose $f_2(x) = (x+3)(x+1)|(x-1)^2|(x-3)$
      i. using a colour other than black, sketch those parts of a graph of $f_2(x)$ that differ from that of $f_0(x)$, using one of the grids below that already shows a graph of $f_0(x)$ in black;
      ii. in the space below write a definition of $f_2(x)$ as a piecewise defined function;
      iii. explain in simple sentences how you determined what the graph of $f_2(x)$ should look like, and then how you used this to give the definition of $f_2(x)$ as a piecewise defined function.

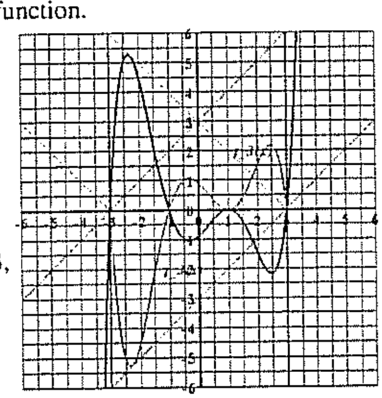


Solution:

iii. The only part of $f_2(x)$ that has absolute value signs around it is the factor $(x-1)^2$. But this factor is squared, hence non-negative already, hence does not change sign when placed inside absolute value signs. Thus, $f_2(x) = f_0(x)$, and their graphs are identical.

ii. The equation in iii. above indicates that in fact $f_2(x)$ need not be piecewise defined at all since all pieces would have the same formula, namely, $f_0(x)$.

   c. suppose $f_3(x) = |(x+3)|(x+1)(x-1)^2|(x-3)|$
      i. using a colour other than black, sketch those parts of a graph of $f_3(x)$ that differ from that of $f_0(x)$, using one of the grids below that already shows a graph of $f_0(x)$ in black;
      ii. in the space below write a definition of $f_3(x)$ as a piecewise defined function;
      iii. explain in simple sentences how you determined what the graph of $f_3(x)$ should look like, and then how you used this to give the definition of $f_3(x)$ as a piecewise defined function.

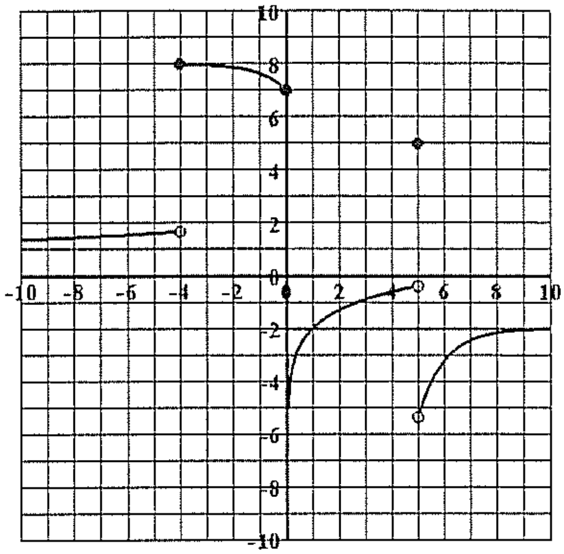


Solution:

iii. This is the most difficult problem. First we note that there are two terms enclosed in absolute value signs: the factors $(x+3)$ and $(x-3)$. How do $|(x+3)|$ and $|(x-3)|$ differ from $(x+3)$ and $(x-3)$ respectively? We draw in the oblique lines for $(x+3)$ and $(x-3)$, and note that graphs for $|(x+3)|$ and $|(x-3)|$ would be "V's" touching the x-axis at $x = -3$ and 3 respectively. That is, $|(x+3)|$ differs from $(x+3)$ only to the left of $x = -3$, and there $|(x+3)| = -(x+3)$. Similarly, $|(x-3)|$ differs from $(x-3)$ only to the left of $x = 3$, and there $|(x-3)| = -(x-3)$. Thus, the x-axis is broken into three segments: $x < -3$, where $|(x+3)|\times|(x-3)| = [-(x+3)]\times[-(x-3)] = (x+3)\times(x-3)$, *i.e.*, no change; $-3 \le x \le 3$, where $|(x+3)|\times|(x-3)| = (x+3)\times[-(x-3)] = -(x+3)\times(x-3)$, *i.e.*, a change by a factor of -1; and $3 < x$, where $|(x+3)|\times|(x-3)| = (x+3)\times(x-3)$, *i.e.*, no change. This shows us that we must reflect the graph of $f_0(x)$ about the x-axis only on the interval [-3,3]. Note that for part of this region the graph of $f_0(x)$ is above the x-axis, and so will be reflected below, and for another part of this region the graph of $f_0(x)$ is below the x-axis, and so will be reflected above.

ii. $$f_3(x) = \begin{cases} f_0(x) & , \quad x < -3 \\ -f_0(x) & , \quad -3 \le x \le 3 \\ f_0(x) & , \quad 3 < x \end{cases}$$

2. Given the graph below of the function $f(x)$ determine all requested limits. No explanation is required, but bonus marks can be earned with good explanations.



a. $\lim_{x\to-\infty} f(x) = 1^+$

We see that at the left edge of the graph the function $f(x)$ values are getting closer and closer to 1, from values that are above 1.

b. $\lim_{x\to-4^-} f(x) = 1.6$

We see that as x is to the left of -4, and getting closer and closer to -4, the values of $f(x)$ are getting closer and closer to about 1.6.

c. $\lim_{x\to-4^+} f(x) = 8$

We see that as x is to the right of -4, and getting closer and closer to -4, the values of $f(x)$ are getting closer and closer to 8.

d. $\lim_{x\to-4} f(x) =$ U or DNE

Since the two half-limits are not of equal value, the whole limit does not exist.

e. $f(-4) = 8$

We see that the vertical line $x = -4$ intersects the graph at $y = 8$.

f. $\lim_{x\to0^-} f(x) = 7$

We see that as x is to the left of 0, and getting closer and closer to 0, the values of $f(x)$ are getting closer and closer to 7.

g. $\lim_{x\to0^+} f(x) = -\infty$

We see that as x is to the left of 0, and getting closer and closer to 0, the values of $f(x)$ are heading down towards $-\infty$.

h. $\lim_{x\to0} f(x) =$ U or DNE

Since the two half-limits are not of equal value, the whole limit does not exist.

i. $f(0) = 7$

We see that the vertical line $x = 0$ intersects the graph at $y = 7$.

j. $\lim_{x\to5^-} f(x) = 0.3$

We see that as x is to the left of 5, and getting closer and closer to 5, the values of $f(x)$ are getting closer and closer to about 0.3.

k. $\lim_{x\to5^+} f(x) = -5.3$

We see that as x is to the right of 5, and getting closer and closer to 5, the values of $f(x)$ are getting closer and closer to about -5.3.

l. $\lim_{x\to5} f(x) =$ U or DNE

Since the two half-limits are not of equal value, the whole limit does not exist.

m. $f(5) =$ U or DNE

We see that the vertical line $x = 5$ does not intersect the graph.

n. $\lim_{x\to\infty} f(x) = -2^-$

We see that at the right edge of the graph the function $f(x)$ values are getting closer and closer to -2, from below -2.

Name: ______________________ Name: ______________________

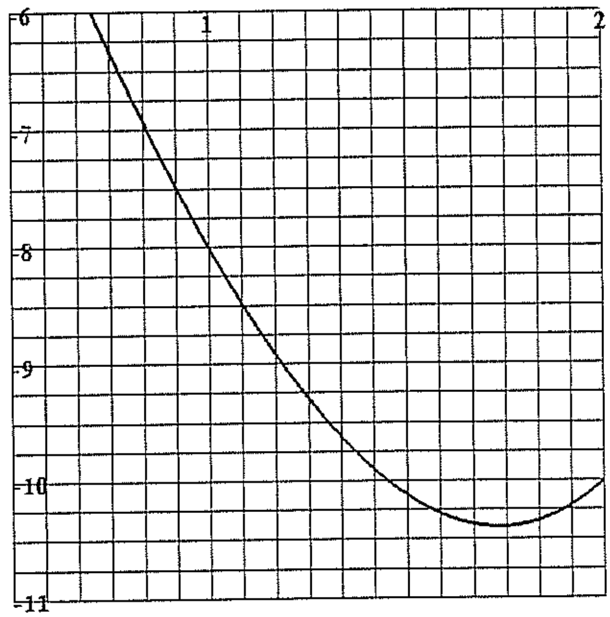
Name: ______________________ Name: ______________________

You will be working in groups of four on this worksheet. Put your name in the first position and that of the other group members elsewhere. All group members must hand in their own copy, but only one will be corrected and the mark will count for all four people. You are to work with the computer, talk, and come up with intelligent answers. From time to time the teacher may intervene to make sure that you are on track.

This material is based on Module 2, Section 6, Rate of Change. This section introduces you to the central concept of Calculus I, Instantaneous Rate of Change, a.k.a. (also known as) Instantaneous Velocity, a.k.a. Slope of a Tangent Line, a.k.a. Derivative. Although we will soon develop quicker and easier methods for computing the derivative, understanding the ideas in this lesson are essential to understanding Calculus I itself. The idea is to move back and forth between the worksheet and the computer, where you can develop the understanding required for answering the worksheet questions.

1. Given a graph of $f(x)$ below (use different colours for a. to d. below) :
   a. sketch a secant line connecting $(1, f(1))$ to $(2, f(2))$ and estimate the slope of this line;
   b. sketch a secant line connecting $(1, f(1))$ to $(1.5, f(1.5))$ and estimate the slope of this line;
   c. sketch a secant line connecting $(1, f(1))$ to $(1.25, f(1.25))$ and estimate the slope of this line;
   d. sketch the line tangent to $f(x)$ at $x = 1$ and estimate the slope of this line;
   e. if x represents time and $f(x)$ represents distance moved by a particle travelling in a straight line, then explain in complete sentences what the slopes determined in a. to d. above represent.

Solution:



In all of the examples below you must show work. Simple answers alone are not enough.

2. Given the function $f(x) = 2x^2 + x - 1$:

   a. $f(1) =$ ______________

   b. $f(2) =$ ______________

   c. slope of line connecting $(1, f(1))$ to $(2, f(2)) =$ ____________________________

   d. $f(1.1) =$ ______________

   e. slope of line connecting $(1, f(1))$ to $(1.1, f(1.1)) =$ ____________________________

   f. $f(1.01) =$ ______________

   g. slope of line connecting $(1, f(1))$ to $(1.01, f(1.01)) =$ ____________________________

   h. $f(1.001) =$ ______________

   i. slope of line connecting $(1, f(1))$ to $(1.001, f(1.001)) =$ ____________________________

   j. $f(1.0001) =$ ______________

   k. slope of line connecting $(1, f(1))$ to $(1.0001, f(1.0001)) =$ ____________________________

   l. slope of the line tangent to $f(x)$ at $(1, f(1)) =$ ____________________________

3. Given the function $f(x) = 2x^2 + x - 1$:

   a. $f(\quad) =$ ____________________________

   b. $f(1+h) =$ ____________________________

   c. $f(1) =$ ____________________________

   d. $f(1+h) - f(1) =$ ____________________________

   e. $\dfrac{f(1+h) - f(1)}{h} =$ ____

   f. $m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h} =$ ____

   g. does your answer to 3. f. agree with your answer to 2. l.? Explain!

   h. determine an equation for the line tangent to $f(x)$ at $x = 1$:

4. Given the function $f(x) = 2x^2 + x - 1$:

a. $f(\quad)$ = ______________________________

b. $f(x+h)$ = ______________________________

c. $f(x)$ = ______________________________

d. $f(x+h) - f(x)$ = ______________________________

e. $\dfrac{f(x+h)-f(x)}{h}$ = ______________________________

f. $f'(x) = \dfrac{df}{dx} = \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$ = ______________________________

= ______________________________

g. $f'(1) =$ ____________

h. does your answer to 4. g. agree with your answers to 3. f. and 2. l.? Explain!

i. determine an equation for the line tangent to $f(x)$ at $x = 1$:

j. $f'(2) =$ ____________

k. determine an equation for the line tangent to $f(x)$ at $x = 2$:

l. $f'(3) =$ ____________

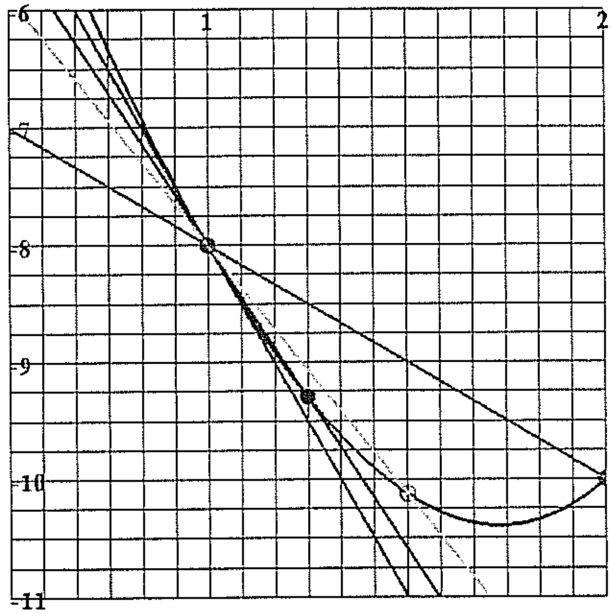
m. determine an equation for the line tangent to $f(x)$ at $x = 3$:

n. $f'(4) =$ ____________

o. determine an equation for the line tangent to $f(x)$ at $x = 4$:

1. Given a graph of $f(x)$ below (use different colours for a. to d. below) :
   a. sketch a secant line connecting $(1, f(1))$ to $(2, f(2))$ and estimate the slope of this line;
   b. sketch a secant line connecting $(1, f(1))$ to $(1.5, f(1.5))$ and estimate the slope of this line;
   c. sketch a secant line connecting $(1, f(1))$ to $(1.25, f(1.25))$ and estimate the slope of this line;
   d. sketch the line tangent to $f(x)$ at $x = 1$ and estimate the slope of this line;
   e. if x represents time and $f(x)$ represents distance moved by a particle travelling in a straight line, then explain in complete sentences what the slopes determined in a. to d. above represent.

Solution:



We note that vertical grid lines cut horizontal units into 12ths and that horizontal grid lines cut vertical units into 4ths. We also note that $(1, f(1)) = (1, -8)$, common to all lines used, lies at a vertex on the grid and so is an accurate point to use when estimating slopes of these lines.

a. **The point $(2, f(2)) = (2, -10)$ also lies at a vertex on the grid so the slope of this secant line (shown in red on the graph) is:**

$$\frac{-10-(-8)}{2-1} = \frac{-2}{1} = -2$$

b. **While the point $(1.5, f(1.5))$ does not seem to lie at a vertex point on the grid, we notice that the point $\left(1\tfrac{5}{12}, f(1\tfrac{5}{12})\right) = \left(1\tfrac{5}{12}, -9\tfrac{3}{4}\right)$ on the green line is a vertex point. Thus, the slope of this secant line is:**

$$\frac{-9\frac{3}{4}-(-8)}{1\frac{5}{12}-1} = \frac{-\frac{7}{4}}{\frac{5}{12}} = -\frac{7}{4}\times\frac{12}{5} = -\frac{21}{5} = -4.2$$

c. **While the point $(1.25, f(1.25))$ does not seem to lie at a vertex point on the grid, we notice that the point $\left(1\tfrac{7}{12}, f(1\tfrac{7}{12})\right) = \left(1\tfrac{7}{12}, -11\right)$ on the blue line is a vertex point. Thus, the slope of this secant line is:**

$$\frac{-11-(-8)}{1\frac{7}{12}-1} = \frac{-3}{\frac{7}{12}} = -3\times\frac{12}{7} = -\frac{36}{7} = -5\tfrac{1}{7} \approx -5.1429$$

d. **The grey tangent line seems to pass through many vertices on the grid, in particular $\left(1\tfrac{4}{12}, -10\right) = \left(1\tfrac{1}{3}, -10\right)$. Thus, the slope of the tangent line is approximately**

$$\frac{-10-(-8)}{1\frac{1}{3}-1} = \frac{-2}{\frac{1}{3}} = -2\times\frac{3}{1} = -6$$

e. **If we think of f as representing the distance moved by a particle travelling in a straight line, and x as time, then the first three slopes are average velocity over increasingly smaller time periods, while the fourth slope, that of the tangent line, represents so called instantaneous velocity at the time $x = 1$.**

In all of the examples below you must show work. Simple answers alone are not enough.

2. Given the function $f(x) = 2x^2 + x - 1$:

   a. $f(1) = 2(1)^2 + (1) - 1 = 2 + 1 - 1 = 2$

   b. $f(2) = 2(2)^2 + (2) - 1 = 2\times4 + 2 - 1 = 9$

   c. slope of line connecting $(1, f(1))$ to $(2, f(2)) = \dfrac{f(2)-f(1)}{2-1} = \dfrac{9-2}{1} = 7$

   d. $f(1.1) = 2(1.1)^2 + (1.1) - 1 = 2\times1.21 + 1.1 - 1 = 2.52$

   e. slope of line connecting $(1, f(1))$ to $(1.1, f(1.1)) = \dfrac{f(1.1)-f(1)}{1.1-1} = \dfrac{2.52-2}{0.1} = 5.2$

f. $f(1.01) = 2(1.01)^2 + (1.01) - 1 = 2\times1.0201 + 1.01 - 1 = 2.0502$

g. slope of line connecting $(1, f(1))$ to $(1.01, f(1.01)) = \dfrac{f(1.01)-f(1)}{1.01-1} = \dfrac{2.0502-2}{0.01} = 5.02$

h. $f(1.001) = 2(1.001)^2 + (1.001) - 1 = 2\times1.002001 + 1.001 - 1 = 2.005002$

i. slope of line connecting $(1, f(1))$ to $(1.001, f(1.001)) = \dfrac{f(1.001)-f(1)}{1.001-1} = \dfrac{2.005002-2}{0.001} = 5.002$

j. $f(1.0001) = 2(1.0001)^2 + (1.0001) - 1 = 2\times1.00020001 + 1.0001 - 1 = 2.00050002$

k. slope of line connecting
$(1, f(1))$ to $(1.0001, f(1.0001)) = \dfrac{f(1.0001)-f(1)}{1.0001-1} = \dfrac{2.00050002-2}{0.0001} = 5.0002$

l. slope of the line tangent to $f(x)$ at $(1, f(1)) = 5$

3. Given the function $f(x) = 2x^2 + x - 1$:

a. $f(\ \ ) = 2(\ \ )^2 + (\ \ ) - 1$

b. $f(1+h) = 2(1+h)^2 + (1+h) - 1 = 2 + 4h + 2h^2 + 1 + h - 1$

c. $f(1) = 2(1)^2 + (1) - 1 = 2 + \quad 1 \quad - 1$

d. $f(1+h) - f(1) = \quad = 4h + 2h^2 \quad + h = 5h + 2h^2 = h(5 + 2h)$

e. $\dfrac{f(1+h)-f(1)}{h} = \dfrac{\not{h}(5+2h)}{\not{h}} = 5 + 2h$

f. $m_{tan} = \lim_{h\to0} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to0}(5+2h) = 5 + 2(0) = 5$

g. does your answer to 3. f. agree with your answer to 2. l.? Explain!

Yes. In 2. we used numeric approximation, using values of h = 1, 0.1, 0.01, 0.001 and 0.0001 and then spotted the pattern, that the slopes of secant lines were slightly larger than 5, but heading closer and closer to 5 as h got closer and closer to 0. In 3. we used symbols and the limit, but we came to the same conclusion: that is, as h gets closer to zero, the slope of the secant lines gets closer to 5.

h. determine an equation for the line tangent to $f(x)$ at $x = 1$:

Since we know the point of tangency is $(1, f(1)) = (1, 2)$, and that the slope of the tangent line is $m = 5$, we use the point-slope of the form of the equation of a line to derive: $\dfrac{y-2}{x-1} = 5 \Leftrightarrow y - 2 = 5(x-1) \Leftrightarrow y = 5x - 3$

4. Given the function $f(x) = 2x^2 + x - 1$:

a. $f(\ \ ) = 2(\ \ )^2 + (\ \ ) - 1$

b. $f(x+h) = 2(x+h)^2 + (x+h) - 1 = 2x^2 + 4xh + 2h^2 + x + h - 1$

c. $f(x) = 2(\ x\ )^2 + (\ x\ ) - 1 = 2x^2 \quad + x \quad - 1$

d. $f(x+h) - f(x) = \quad = + 4xh + 2h^2 \quad + h = h(4x + 2h + 1)$

e. $\frac{f(x+h)-f(x)}{h} = \frac{\not{h}(4x+2h+1)}{\not{h}} = 4x+2h+1$

f. $f'(x) = \frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(4x+2h+1) = 4x+2(0)+1 = 4x+1$

g. $f'(1) = 4(1) + 1 = 5$

h. does your answer to 4. g. agree with your answers to 3. f. and 2. l.? Explain!

Yes, all of these answers agree. They are all variations on the same theme - if we choose secant lines that all pass through the same fixed point, and move the second point of contact closer and closer to that point, the secant line will become more and more like a tangent line. This last method has the great advantage that to compute the value of the slope of the tangent line at any value of x, once we have finished step f., we need only substitute in the appropriate value of x.

i. determine an equation for the line tangent to $f(x)$ at $x = 1$:

Note that this is the same question as 3. h. above, and hence has the same answer. Since we know the point of tangency is $(1, f(1)) = (1, 2)$, and that the slope of the tangent line is $m = 5$, we use the point-slope of the form of the equation of a line to derive: $\frac{y-2}{x-1} = 5 \Leftrightarrow y-2 = 5(x-1) \Leftrightarrow y = 5x-3$

j. $f'(2) = 4(2) + 1 = 9$

k. determine an equation for the line tangent to $f(x)$ at $x = 2$:

Note that this is the same question as 4. i. above, except at a different value, and hence has a similar answer. Since we know the point of tangency is $(2, f(2)) = (2, 9)$, and that the slope of the tangent line is $m = 9$, we use the point-slope of the form of the equation of a line to derive: $\frac{y-9}{x-2} = 9 \Leftrightarrow y-9 = 9(x-2) \Leftrightarrow y = 9x-9$

l. $f'(3) = 4(3) + 1 = 13$

m. determine an equation for the line tangent to $f(x)$ at $x = 3$:

Note that this is the same question as 4. i. above, except at a different value, and hence has a similar answer. Since we know the point of tangency is $(3, f(3)) = (3, 9)$, and that the slope of the tangent line is $m = 9$, we use the point-slope of the form of the equation of a line to derive:

$$\frac{y-20}{x-3} = 13 \Leftrightarrow y-20 = 13(x-3) \Leftrightarrow y = 13x-19$$

n. $f'(4) = 4(4) + 1 = 17$

o. determine an equation for the line tangent to $f(x)$ at $x = 4$:

Note that this is the same question as 4. i. above, except at a different value, and hence has a similar answer. Since we know the point of tangency is $(4, f(4)) = (4, 35)$, and that the slope of the tangent line is $m = 17$, we use the point-slope of the form of the equation of a line to derive:

$$\frac{y-35}{x-4} = 17 \Leftrightarrow y-35 = 17(x-4) \Leftrightarrow y = 17x-33$$

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This material is based on Module 2, Section 7, The Derivative as a Function. Note that this section: introduces a few new words (derivative, differentiable, and differentiability); introduces some new notations ($f'(x)$, $\frac{df}{dx}$ $D_x f(x)$); helps you to practice computing $f'(x)$ using the definition (Newton's Quotient); and, discusses the relationship between differentiability and continuity. This worksheet will test your understanding of these. The idea is to move back and forth between the worksheet and the computer, where you can develop the understanding required for answering the worksheet questions.

1. The derivative of a given function $f(x)$, be it written as $f'(x)$ or $\frac{df}{dx}$, is a new function, related to the original.

   a. If $x(t)$ represents position at time t, then $x'(t) = \frac{dx}{dt}$ represents ______________________

   b. If we are looking at the graph of $f(x)$, then $f'(x) = \frac{df}{dx}$ represents ______________________

   and this explains why $f'(x)$ equals a constant value if and only if ______________________

2. By definition a function $f(x)$ is differentiable at $x = a$ if and only if the derivative of $f(x)$ exists at $x = a$. Looking at the graph of the function we say that it is differentiable at the point $(a, f(a))$ if it is "smooth" there or that the graph does not have a "sharp" point there. In the previous lesson we saw that smooth meant that if we zoomed in, then the graph eventually looks like a straight line. What does the existence of $f'(a)$ have to do with zooming in creating a straight line? Explain in your own words.

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3. Given the function $f(x) = \dfrac{1}{2x+1}$

   a. On the back of this sheet go through **all** steps, as laid out in the lesson and examples, of computing $f'(x)$ using Newton's Quotient.

   b. Will $f'(x)$ exist for all values of x? Explain why your answer was predictable **before** $f'(x)$ was known.

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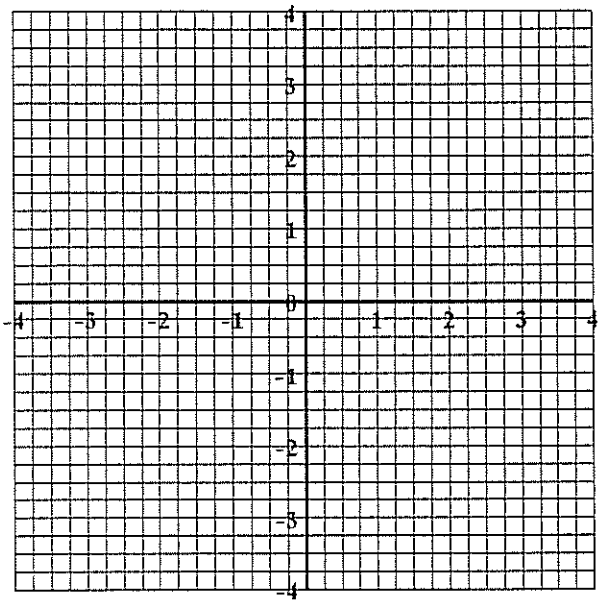
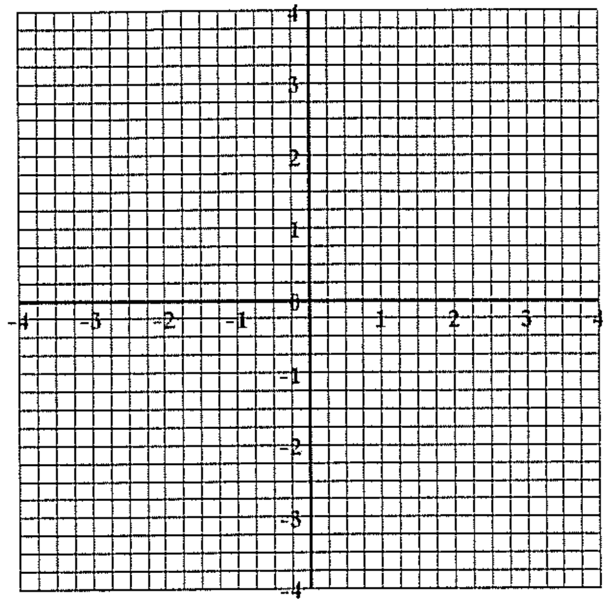
   c. On the back of this sheet show how you determine the y coordinate of the point on the graph of $f(x)$ at $x = -1$ and the slope of the tangent line to $f(x)$ at this point. Use your answers to compute the equation of the tangent line to $f(x)$ at $x = -1$.

4. Claim: Every differentiable function is continuous, but not every continuous function is differentiable. For two different functions that are continuous but not differentiable, not taken from the examples or the lesson, sketch graphs of such functions and explain.



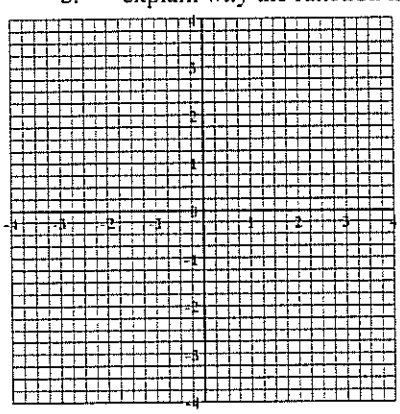
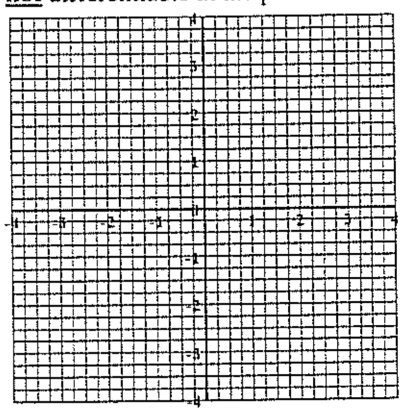
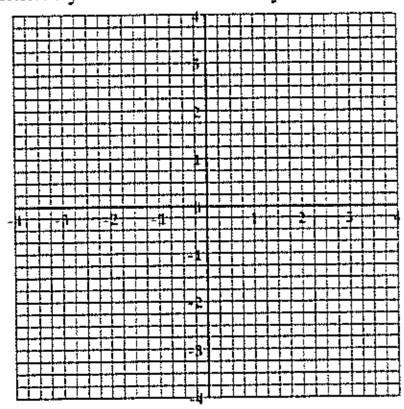


5. In the branch of mathematics called logic we refer to a statement of the form "if A then B" as an "if-then" statement. For any such statement there are three related ones: the converse ("if B then A"); the inverse ("if not-A then not-B"); and the "contrapositive" ("if not-B then not-A").

For the if-then statement, "if a function is differentiable, then it is continuous" we have seen that the converse statement, "if a function is continuous, then it is differentiable" is **not** true (you have just shown two examples in 3. above). However, the contrapositive statement "if a function is not continuous, then it is not differentiable" is equivalent to the original statement, and hence true.

For each of the three types of discontinuities (removable, a.k.a. misplaced point, jump a.k.a. vertical jump, infinite a.k.a. vertical asymptote) carry out the following steps:

a. sketch a function with such a discontinuity, as well as three secant lines approaching the point of discontinuity;
b. explain why the function is **not** differentiable at the point of discontinuity with reference to your sketch.

1. The derivative of a given function $f(x)$, be it written as $f'(x)$ or $\dfrac{df}{dx}$, is a new function, related to the original.

   a. If $x(t)$ represents position at time t, then $x'(t) = \dfrac{dx}{dt}$ represents: **instantaneous velocity at time t**

   b. If we are looking at the graph of $f(x)$, then $f'(x) = \dfrac{df}{dx}$ represents: **the slope of a tangent line**

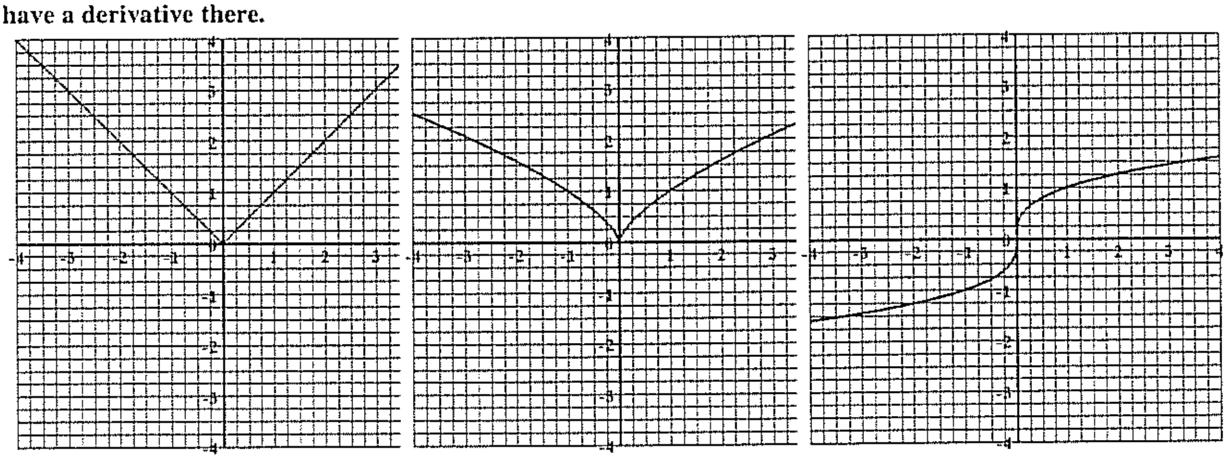
   and this explains why $f'(x)$ equals a constant value if and only if: **$f(x)$ is linear, *i.e.*, has constant slope**

2. By definition a function $f(x)$ is differentiable at $x = a$ if and only if the derivative of $f(x)$ exists at $x = a$. Looking at the graph of the function we say that it is differentiable at the point $(a, f(a))$ if it is "smooth" there or that the graph does not have a "sharp" point there. In the previous lesson we saw that smooth meant that if we zoomed in, then the graph eventually looks like a straight line. What does the existence of $f'(a)$ have to do with zooming in creating a straight line? Explain in your own words.

Solution:
The derivative, $f'(a)$, exists if the limit of secant line slopes exists, *i.e.*, if we can establish a pattern in a sequence of secant line slopes, and we call that pattern or eventual value the slope of the tangent line. A curve appears to be smooth if we can zoom in and eventually the curve appears to become a straight line, the tangent line. The curve is not smooth if there is a "sudden change in direction" or a "sharp point" that remains when we zoom in. Finding an "eventual value" requires us to choose ever smaller values of "h", which is the same as zooming in, so that the eventual value is the slope of the straight line that a smooth curve eventually "becomes".

3. In this section it is asserted that every differentiable function is continuous, but not every continuous function is differentiable. Sketch the graphs and provide an accompanying explanation for two different functions that are continuous but not differentiable.

Solution:
Three relatively simple functions that are continuous (everywhere), but not differentiable (everywhere) are $f(x) = |x|$, $g(x) = x^{2/3}$, and $h(x) = x^{1/3}$. From graphs of these functions, shown on the side, we see that graphs of these functions can easily be drawn without lifting the pencil at all. However, the first two graphs have a sharp point, *i.e.*, one that is not smooth, at $x = 0$. That is, if we zoomed in on these graphs so as to look only at values of x that are ever more close to $x = 0$, the graph never looks like a non-vertical line of a fixed slope. The third graph has a vertical tangent line at $x = 0$, hence has "infinite slope" there, and so does not actually have a derivative there.



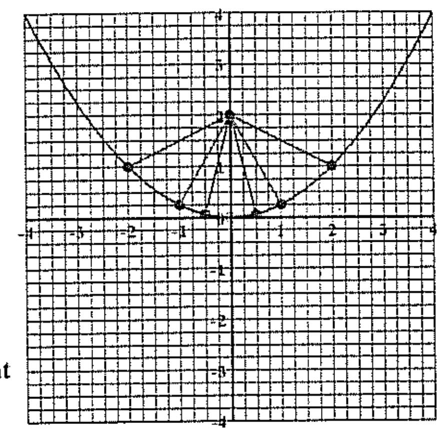
$f(x)$ $g(x)$ $h(x)$

4. In the branch of mathematics called logic we refer to a statement of the form "if A then B" as an "if-then" statement. For any such statement there are three related ones: the converse ("if B then A"); the inverse ("if not-A then not-B"); and the "contrapositive" ("if not-B then not-A"). For the if-then statement, "if a function is differentiable, then it is continuous" we have seen that the converse statement, "if a function is continuous, then it is differentiable" is <u>not</u> true (you have just shown two examples in 3. above). However, the contrapositive statement "if a function is not continuous, then it is not differentiable" is equivalent to the original statement, and hence true. For each of the three types of discontinuities (removable (misplaced point), jump (vertical jump), infinite (vertical asymptote) carry out the following steps:
   a. sketch a function with such a discontinuity, as well as three secant lines approaching the point of discontinuity;
   b. explain why the function is <u>not</u> differentiable at the point of discontinuity with reference to your sketch.

Solutions:

Removable Discontinuity

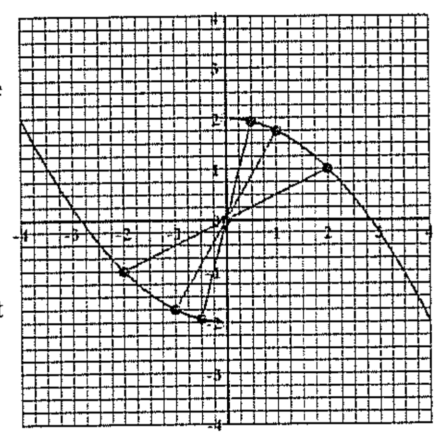
In the graph of $f(x)$ sketched at the right we note that there is a removable discontinuity at $x = 0$. In this example, when we connect a point on the graph with a value of x close to 0, to the point $(0, f(0)) = (0, 2)$, we clearly see that the secant line thus drawn has a slope that is positive or negative, depending upon which side of $x = 0$ the point lies. Further, as we draw many such secant lines, with the second point drawing closer and closer to $x = 0$, we note that the absolute value of the slope grows larger and larger without bound, hence approaching ∞ as the x-coordinate of the second point approaches $x = 0$. Thus, this function is not differentiable at the point of discontinuity since it has no tangent line, hence no slope of a tangent line, at $x = 0$.



$f(x)$

Jump Discontinuity

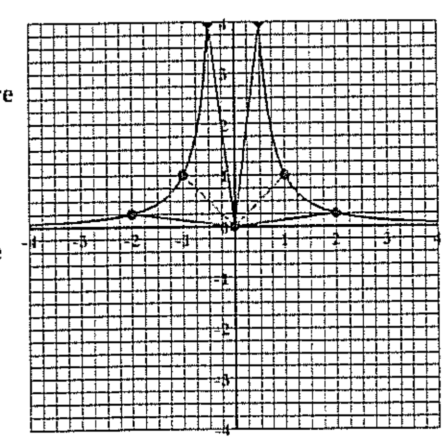
In the graph of $f(x)$ sketched at the right we note that there is a jump discontinuity at $x = 0$. In this example, when we connect a point on the graph with a value of x close to 0, to the point $(0, f(0)) = (0, 0)$, we clearly see that the secant line thus drawn has a slope that is positive. As we draw many such secant lines, with the second point drawing closer and closer to $x = 0$, we note that the value of the slope grows larger and larger without bound, hence approaching ∞ as the x-coordinate of the second point approaches $x = 0$. Thus, this function is not differentiable at the point of discontinuity since it has no tangent line, hence no slope of a tangent line, at $x = 0$.



$f(x)$

Infinite Discontinuity

In the graph of $f(x)$ sketched at the right we note that there is an infinite discontinuity at $x = 0$. Usually at an infinite discontinuity there is no point. When there is no point, there can be no tangent line (no point of tangency), hence no slope of the tangent line, hence no derivative. However, suppose there was a point, as shown on the graph, (0,0). We note that depending upon which side we draw successive secant lines from, the slopes of the secant lines appear to be approaching either $-\infty$ or $+\infty$. Thus, there is no tangent line, and no slope of a tangent line, hence no derivative, at the point of discontinuity.



$f(x)$

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You will be working in pairs on this worksheet. Put your name in the first position and that of your partner in the other position. Both pair members must hand in their own copy, but only one will be corrected and the mark will count for both people. You are to work with the computer, talk, and come up with intelligent answers. From time to time the teacher may intervene to make sure that you are on track.

This material is based on Module 3, Section 1, Rules of Differentiation as well as Section 2, Implicit and Logarithmic Differentiation. The idea is to move back and forth between the worksheet and the computer, where you can develop the understanding required for answering the worksheet questions.

1. The theory of relativity predicts that an object whose mass is m_0 when it is at rest will appear heavier when moving at speeds near the speed of light, c. When the object is moving at speed v, its mass m is given by:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

   a. Determine $\frac{dm}{dv}$. (If you wish to use the Derivative Calculator - Module 3, Section 1, Page VIII, bottom - to check your answer then you must make m a function of x instead of v, or use the variable button to choose v as your variable).

   b. In terms of Physics, what does $\frac{dm}{dv}$ tell you?

Solution:

2. A particle is moving along the x-axis, where x is in centimetres. Its velocity, v, in cm/sec, when it is at the point with coordinate x is given by: $v = x^2 + 3x - 2$. Determine the acceleration of the particle when it is at the point $x = 2$. Give units in your answer.

Solution:

3. A function $f(x)$ is said to have a *zero of multiplicity* m at $x = a$ if:

$$f(x) = (x - a)^m h(x), \text{ with } h(a) \neq 0$$

Explain why a function having a zero of multiplicity m at $x = a$ satisfies the following equation:

$$f^{(p)}(a) = \left.\frac{d^p f(x)}{dx^p}\right|_{x=a} = 0, \text{ for } p = 1, 2, \ldots, m-1$$

Solution:

4. The equation $x^2 + xy + y^2 = 7$ defines a curve.
   a. Determine the two points where this curve crosses the x-axis and show that the tangent lines to the curve at these points are parallel. What is the common slope of these tangent lines?
   b. Determine points on this curve where the tangent is parallel to the x-axis.
   c. Determine points on this curve where the tangent is parallel to the y-axis. (Note that in this case $\frac{dy}{dx}$ is undefined, but $\frac{dx}{dy}$ is. What value does $\frac{dx}{dy}$ have at these points?)

Solution:

1. The theory of relativity predicts that an object whose mass is m_0 when it is at rest will appear heavier when moving at speeds near the speed of light, c. When the object is moving at speed v, its mass m is given by:

$$m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

   a. Determine $\frac{dm}{dv}$. (If you wish to use the Derivative Calculator - Module 3, Section 1, Page VIII, bottom - to check your answer then you must make m a function of x instead of v, or use the variable button to choose v as your variable).

   b. In terms of Physics, what does $\frac{dm}{dv}$ tell you?

Solution:

a.

$m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}=m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}$	Use Algebra to rewrite function to make Calculus easier.
$\frac{dm}{dv}=\frac{dm_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}}{dv}=m_0\frac{d\left(1-\frac{v^2}{c^2}\right)^{-1/2}}{dv}$	Multiplication by a Constant Rule
$=m_0\frac{d\left(1-\frac{v^2}{c^2}\right)^{-1/2}}{d\left(1-\frac{v^2}{c^2}\right)}\times\frac{d\left(1-\frac{v^2}{c^2}\right)}{dv}$	Chain Rule
$=m_0\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left[\frac{d1}{dv}-\frac{d\left(\frac{v^2}{c^2}\right)}{dv}\right]$	Power and Difference Rules
$=m_0\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left[0-\left(\frac{1}{c^2}\right)\frac{dv^2}{dv}\right]$	Constant and Multiplication by a Constant Rule
$=m_0\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left[-\left(\frac{1}{c^2}\right)2v\right]$	Power Rule
$=\left(\frac{m_0v}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}=\frac{\left(\frac{m_0v}{c^2}\right)}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^3}$	Algebra to simplify
$=\frac{\left(\frac{m_0v}{c^2}\right)}{\left(\sqrt{\frac{c^2-v^2}{c^2}}\right)^3}=\left(\frac{m_0v}{c^2}\right)\times\frac{c^3}{\left(\sqrt{c^2-v^2}\right)^3}=\frac{m_0vc}{\left(\sqrt{c^2-v^2}\right)^3}$	Algebra to simplify

b. The first thing we notice is that $\frac{dm}{dv}$ is positive, indicating that as v increases, so does m. If we take a closer look we can also see that $\frac{dm}{dv}$ increases as v approaches c from the left (can you explain why such a limit is only from the left?). We deduce this from the fact that the denominator decreases towards 0 as v approaches c, which tends to make the fraction larger. At the same time the numerator increases, which also tends to make the fraction larger. Thus, as you near the speed of light, your mass is increasing faster and faster.

2. A particle is moving along the x-axis, where x is in centimetres. Its velocity, v, in cm/sec, when it is at the point with coordinate x is given by: $v = x^2 + 3x - 2$. Determine the acceleration of the particle when it is at the point $x = 2$. Give units in your answer.

Solution:

We are given that $\frac{dx}{dt} = v = x^2 + 3x - 2$ and we are asked to compute $\left.\frac{dv}{dt}\right|_{x=2}$.

We note that according to the Chain Rule, $\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$.

But v is a polynomial function of x, so $\frac{dv}{dx} = \frac{d\left(x^2+3x-2\right)}{dx} = \frac{d\,x^2}{dx} + \frac{d3x}{dx} - \frac{d2}{dx} = 2x + 3\frac{d\,x}{dx} - 0 = 2x+3$.

Thus, according to the Chain Rule, $\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = (2x+3) \times \left(x^2+3x-2\right)$ and

$$\left.\frac{dv}{dt}\right|_{x=2} = \left[(2x+3)\times\left(x^2+3x-2\right)\right]_{x=2} = (2(2)+3)\times\left((2)^2+3(2)-2\right) = 7\times 8 = 56$$

This would be measured in cm/(sec)^2.

3. A function $f(x)$ is said to have a *zero of multiplicity m* at $x = a$ if:

$$f(x) = (x-a)^m\, h(x), \text{ with } h(a) \neq 0$$

Explain why a function having a zero of multiplicity m at $x = a$ satisfies the following equation:

$$f^{(p)}(a) = \left.\frac{d^p f(x)}{dx^p}\right|_{x=a} = 0, \text{ for } p = 1, 2, \ldots, m-1$$

Solution:

One way to begin is to try calculating, as much as we can, the sequence of derivatives, $f'(x), f''(x), f'''(x), \ldots, f^{(m-1)}(x)$. Since we do not actually know the function $h(x)$, the best we will be able to do is have answers that include $h'(x), h''(x), h'''(x), \ldots, h^{(m-1)}(x)$.

$$\begin{aligned}
f(x) &= (x-a)^m\, h(x)\\
f'(x) &= [(x-a)^m h(x)]' = [(x-a)^m]'h(x) + (x-a)^m h'(x) = m(x-a)^{m-1}(1)h(x) + (x-a)^m h'(x)\\
&= (x-a)^{m-1}[mh(x) + (x-a)h'(x)]\\
f''(x) &= [(x-a)^{m-1}]'[mh(x)+(x-a)h'(x)] + (x-a)^{m-1}[mh(x)+(x-a)h'(x)]'\\
&= (m-1)(x-a)^{m-2}(1)[mh(x)+(x-a)h'(x)] + (x-a)^{m-1}[mh'(x)+\{(x-a)'h'(x)+(x-a)h''(x)\}]\\
&= (m-1)(x-a)^{m-2}[mh(x)+(x-a)h'(x)] + (x-a)^{m-1}[mh'(x)+(1)h'(x)+(x-a)h''(x)]\\
&= (x-a)^{m-2}\{(m-1)[mh(x)+(x-a)h'(x)] + (x-a)[mh'(x)+h'(x)+(x-a)h''(x)]\}\\
&= (x-a)^{m-2}\{(m-1)mh(x) + (m-1)(x-a)h'(x) + m(x-a)h'(x) + (x-a)h'(x) + (x-a)^2 h''(x)\}\\
&= (x-a)^{m-2}\{(m-1)mh(x) + [(m-1)+m+1](x-a)h'(x) + (x-a)^2 h''(x)\}\\
&= (x-a)^{m-2}[(m-1)mh(x) + 2m(x-a)h'(x) + (x-a)^2 h''(x)]
\end{aligned}$$

While there are many interesting patterns that we might observe here, we will focus on the most obvious ones.

The patterns that we observe are that: (1) from the first derivative we can factor out $(x-a)^{m-1}$, from the second derivative we can factor out $(x-a)^{m-2}$, so we deduce that from the p^{th} derivative we can factor out $(x-a)^{m-p}$; (2) when we factor out $(x-a)^{m-1}$ from the first derivative what is left is a constant times $h(x)$ plus a constant times $(x-a)$ times $h'(x)$, when we factor out $(x-a)^{m-2}$ from the second derivative, what is left is a constant times $h(x)$ plus a constant times $(x-a)$ times $h'(x)$, plus a constant times $(x-a)^2$ times $h''(x)$, so we deduce that when we factor out $(x-a)^{m-p}$ from the p^{th} derivative, what is left is a constant times $h(x)$ plus a constant times $(x-a)$ times $h'(x)$, plus a constant times $(x-a)^2$ times $h''(x)$, etc., until a constant times $(x-a)^p$ times $h^{(p)}(x)$. Thus, we obtain: $f^{(p)}(x) = (x-a)^{m-p}[c_0 h(x) + c_1(x-a)h'(x) + c_2(x-a)^2 h''(x) + \ldots + c_p(x-a)^p h^{(p)}(x)]$, where the c_i are the constant coefficients, none of which is 0.

(With some effort we could discern patterns for each of the constant coefficients - I leave this to you to investigate, but it is not necessary for the purpose of solving the current problem.)

Thus, as long as $m-p > 0$, $f^{(p)}(a) = 0$. We note that $f^{(m)}(a) = c_0 h(a) + 0 \neq 0$ (given).

4. The equation $x^2 + xy + y^2 = 7$ defines a curve.
   a. Determine the two points where this curve crosses the x-axis and show that the tangent lines to the curve at these points are parallel. What is the common slope of these tangent lines?
   b. Determine points on this curve where the tangent is parallel to the x-axis.
   c. Determine points on this curve where the tangent is parallel to the y-axis. (Note that in this case $\frac{dy}{dx}$ is undefined, but $\frac{dx}{dy}$ is. What value does $\frac{dx}{dy}$ have at these points?)

Solution:

a. All points on the x-axis satisfy the the equation $y = 0$. Thus, to determine where the curve $x^2 + xy + y^2 = 7$ intersects the x-axis we substitute in 0 for y and then solve the resulting equation for x:

$$x^2 + x(0) + (0)^2 = 7 \Rightarrow x^2 = 7 \Rightarrow x = \pm\sqrt{7}$$

Now, to determine the slopes of the tangent lines at these points, $\left(-\sqrt{7},0\right), \left(\sqrt{7},0\right)$, we must first use implicit differentiation to determine $\frac{dy}{dx}$:

$$x^2 + xy + y^2 = 7 \Rightarrow \frac{d\left(x^2+xy+y^2\right)}{dx} = \frac{d7}{dx} \Leftrightarrow \frac{dx^2}{dx} + \frac{dxy}{dx} + \frac{dy^2}{dx} = 0 \Leftrightarrow 2x + \left[\left(\frac{dx}{dx}\right)y + x\left(\frac{dy}{dx}\right)\right] + \frac{dy^2}{dy}\times\frac{dy}{dx} = 0$$

$$\Leftrightarrow 2x + \left[(1)y + x\left(\frac{dy}{dx}\right)\right] + 2y\times\frac{dy}{dx} = 0 \Leftrightarrow 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \Leftrightarrow (x+2y)\frac{dy}{dx} = -(2x+y) \Leftrightarrow \frac{dy}{dx} = -\frac{2x+y}{x+2y}$$

Now, substituting in the coordinates of each of the x-intercept points:

$$\left.\frac{dy}{dx}\right|_{\substack{x=-\sqrt{7}\\ y=0}} = -\frac{2\left(-\sqrt{7}\right)+(0)}{\left(-\sqrt{7}\right)+2(0)} = 2, \left.\frac{dy}{dx}\right|_{\substack{x=\sqrt{7}\\ y=0}} = -\frac{2\left(\sqrt{7}\right)+(0)}{\left(\sqrt{7}\right)+2(0)} = 2$$

Since the slope (derivative) is the same at both points, we know that the two tangent lines are parallel. The slope, as seen above, is 2, for both tangent lines.

b. A line that is parallel to the x-axis has a slope of 0. Thus, the derivative formula that we computed above should be set equal to 0, and using the information gleaned from that, we return to the original equation and attempt to solve for all possible values of x & y:

$$\frac{dy}{dx} = -\frac{2x+y}{x+2y} = 0 \Leftrightarrow 2x + y = 0 \Leftrightarrow y = -2x \Rightarrow x^2 + x(-2x) + (-2x)^2 = 7 \Leftrightarrow 3x^2 = 7 \Leftrightarrow x^2 = \frac{7}{3} \Leftrightarrow x = \pm\sqrt{\frac{7}{3}}, y = \mp 2\sqrt{\frac{7}{3}}$$

Note that the denominator of $\frac{dy}{dx}$ is not zero at either of these two points, so the derivative is really 0 there.

That is, the tangent line to this curve is parallel to the x-axis at the points: $\left(-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}}\right), \left(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}\right)$

c. Approach 1: A line that is parallel to the y-axis has an undefined (infinite) slope. Thus, if a tangent line were vertical (*i.e.*, parallel to the y-axis), then the derivative would be undefined. This would probably mean that the derivative has a denominator that is 0 (and a numerator that is not 0).

$$\frac{dy}{dx} = -\frac{2x+y}{x+2y} = U \Leftrightarrow x + 2y = 0 \Leftrightarrow x = -2y \Rightarrow (-2y)^2 + (-2y)y + y^2 = 7 \Leftrightarrow 3y^2 = 7 \Leftrightarrow y^2 = \frac{7}{3} \Leftrightarrow y = \pm\sqrt{\frac{7}{3}}, x = \mp 2\sqrt{\frac{7}{3}}$$

Note that the numerator of $\frac{dy}{dx}$ is not zero at either of these points, so at these points the slope is undefined as predicted, which should mean that the tangent lines are parallel to the y-axis.

Approach 2: If we think of the equation defining x as a function of y, then $\frac{dx}{dy}$ represents the "slope", now measured as $\frac{\text{run}}{\text{rise}}$, of a line tangent to this function. In this context, what we normally think of as vertical lines, *i.e.*, lines parallel to the y-axis, whose equation is of the form $x = c$ where c is some constant, actually have a "slope" of 0. We could differentiate the equation with respect to y this time and determine a formula for $\frac{dx}{dy}$, but there is an easier way that also illustrates a general theorem, so we shall do that instead.

$$1=\frac{dx}{dx}=\frac{dx}{dy}\times\frac{dy}{dx}\Rightarrow\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{dx}{dy}$$

Thus, using our previous computations we obtain:

$$\frac{dy}{dx}=-\frac{2x+y}{x+2y}\Rightarrow\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{\left(-\frac{2x+y}{x+2y}\right)}=-\frac{x+2y}{2x+y}$$

Thus, having slope of 0, would be the same as asking for $(x+2y)=0$, which from above we know led to the points $\left(-2\sqrt{\frac{7}{3}},\sqrt{\frac{7}{3}}\right),\left(2\sqrt{\frac{7}{3}},-\sqrt{\frac{7}{3}}\right)$.

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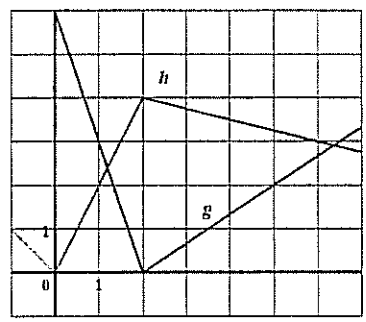
In the problems below, show each and every step. Explain, in sentences in English, what you are doing as if you were writing a solution manual for other students. If you need more space, use the back of the sheet.

1. A table of values of f, g, f' and g' is given below on the right:
   a. if $h(x) = f(g(x))$, determine $h'(1)$;
   b. if $j(x) = g(f(x))$, determine $j'(1)$;
   c. if $k(x) = f(f(x))$, determine $k'(1)$;
   d. if $l(x) = g(g(x))$, determine $l'(1)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

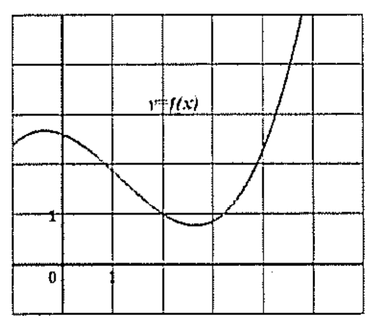
Solutions:

2. If g and h are the functions whose graphs are shown on the right, let $u(x) = h(g(x))$, $v(x) = g(h(x))$, and $w(x) = g(g(x))$. Determine each of the derivatives indicated below, if it exists. If it does not exist, explain why it does not exist.
   a. $u'(1)$ b. $v'(1)$ c. $w'(1)$



Solutions:

3. If f is the function whose graph is shown on the right, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each of the derivatives indicated below.
   a. $h'(2)$ b. $g'(2)$



Solutions:

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   a. if $h(x) = f(g(x))$, determine $h'(1)$;
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   c. if $k(x) = f(f(x))$, determine $k'(1)$;
   d. if $l(x) = g(g(x))$, determine $l'(1)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

Solutions:

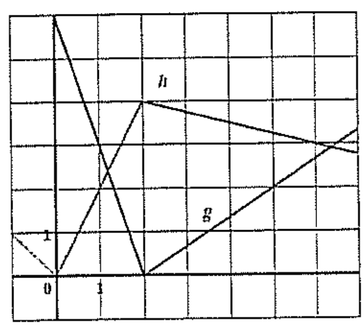
a.
$$h'(x) = \frac{d\,h(x)}{dx} = \frac{d\,f(g(x))}{dx} = \frac{d\,f(g(x))}{dg(x)} \times \frac{d\,g(x)}{dx}$$
$$h'(1) = \left.\frac{d\,f(g(x))}{dg(x)}\right|_{x=1} \times \left.\frac{d\,g(x)}{dx}\right|_{x=1} = f'(g(1)) \times g'(1) = f'(2) \times 6 = 5 \times 6 = 30$$

b.
$$j'(x) = \frac{d\,j(x)}{dx} = \frac{d\,g(f(x))}{dx} = \frac{d\,g(f(x))}{df(x)} \times \frac{d\,f(x)}{dx}$$
$$j'(1) = \left.\frac{d\,g(f(x))}{df(x)}\right|_{x=1} \times \left.\frac{d\,f(x)}{dx}\right|_{x=1} = g'(f(1)) \times f'(1) = g'(3) \times 4 = 9 \times 4 = 36$$

c.
$$k'(x) = \frac{d\,k(x)}{dx} = \frac{d\,f(f(x))}{dx} = \frac{d\,f(f(x))}{df(x)} \times \frac{d\,f(x)}{dx}$$
$$k'(1) = \left.\frac{d\,f(f(x))}{df(x)}\right|_{x=1} \times \left.\frac{d\,f(x)}{dx}\right|_{x=1} = f'(f(1)) \times f'(1) = f'(3) \times 4 = 7 \times 4 = 28$$

d.
$$l'(x) = \frac{d\,l(x)}{dx} = \frac{d\,g(g(x))}{dx} = \frac{d\,g(g(x))}{dg(x)} \times \frac{d\,g(x)}{dx}$$
$$l'(1) = \left.\frac{d\,g(g(x))}{dg(x)}\right|_{x=1} \times \left.\frac{d\,g(x)}{dx}\right|_{x=1} = g'(g(1)) \times g'(1) = g'(2) \times 6 = 7 \times 6 = 42$$

2. If g and h are the functions whose graphs are shown on the right, let $u(x) = h(g(x))$, $v(x) = g(h(x))$, and $w(x) = g(g(x))$. Determine each of the derivatives indicated below, if it exists. If it does not exist, explain why it does not exist.
   a. $u'(1)$ b. $v'(1)$ c. $w'(1)$



Solutions:

Before we begin, we note from the graphs of g and h that they are both piecewise-linear functions. Thus, on each piece the derivative is simply the slope of that line segment. However, at each of the "seams", where the lines meet, these are "sharp" points at which there is no derivative (no tangent line, hence no tangent line slopes).

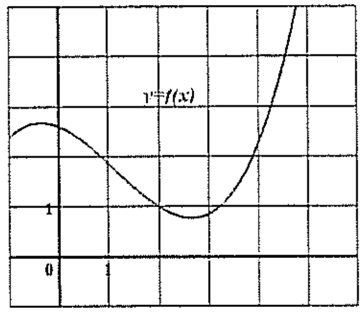
a.
$$u'(x) = \frac{d\,u(x)}{dx} = \frac{d\,h(g(x))}{dx} = \frac{d\,h(g(x))}{dg(x)} \times \frac{d\,g(x)}{dx}$$
$$u'(1) = \left.\frac{d\,h(g(x))}{dg(x)}\right|_{x=1} \times \left.\frac{d\,g(x)}{dx}\right|_{x=1} = h'(g(1)) \times g'(1) = h'(3) \times \left(\frac{-6}{2}\right) = \left(\frac{-1}{4}\right) \times (-3) = \frac{3}{4}$$

b.
$$v'(x) = \frac{d\,v(x)}{dx} = \frac{d\,g(h(x))}{dx} = \frac{d\,g(h(x))}{dh(x)} \times \frac{d\,h(x)}{dx}$$
$$v'(1) = \left.\frac{d\,g(h(x))}{dh(x)}\right|_{x=1} \times \left.\frac{d\,h(x)}{dx}\right|_{x=1} = g'(h(1)) \times h'(1) = g'(2) \times \left(\frac{4}{2}\right) = U \times 2 = U$$

That is, $v'(1)$ does not exist, because g has a sharp point at $h(1) = 2$.

c.
$$w'(x) = \frac{d\,w(x)}{dx} = \frac{d\,g(g(x))}{dx} = \frac{d\,g(g(x))}{dg(x)} \times \frac{d\,g(x)}{dx}$$
$$w'(1) = \left.\frac{d\,g(g(x))}{dg(x)}\right|_{x=1} \times \left.\frac{d\,g(x)}{dx}\right|_{x=1} = g'(g(1)) \times g'(1) = g'(3) \times \left(\frac{-6}{2}\right) = \left(\frac{2}{3}\right) \times (-3) = -2$$

3. If f is the function whose graph is shown on the right, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each of the derivatives indicated below.
   a. $h'(2)$ b. $g'(2)$



Solutions:

$$h'(x) = \frac{d\,h(x)}{dx} = \frac{d\,f(f(x))}{dx} = \frac{d\,f(f(x))}{df(x)} \times \frac{d\,f(x)}{dx}$$

a. $$h'(2) = \left.\frac{d\,f(f(x))}{df(x)}\right|_{x=2} \times \left.\frac{d\,f(x)}{dx}\right|_{x=2} = f'(f(2)) \times f'(2)$$

$$\cong f'(1) \times \left(\frac{-2}{3}\right) = (-1) \times \left(\frac{-2}{3}\right) = \frac{2}{3}$$

Note: The derivatives, $f'(2)$ and $f'(1)$, are estimated by placing a transparent plastic ruler in the position of tangent line at $(2, f(2))$ and then at $(1, f(1))$ and computing the slope. Thus, the tangent at $(2, f(2))$ appears to pass through (-1,3) and (5,-1), so the "rise" = (-1 - 3) = -4, "run" = (5 - (-1)) = 6, so $f'(2) \cong -(4/6) = -2/3$. Similarly, the tangent at $(1, f(1))$ appears to pass through (2,1) and (3,0), so the "rise" = (0 - 1) = -1, "run" = (3 - 2) = 1, so $f'(1) \cong -(1/1) = -1$.

b.
$$g'(x) = \frac{d\,g(x)}{dx} = \frac{d\,f(x^2)}{dx} = \frac{d\,f(x^2)}{dx^2} \times \frac{d\,x^2}{dx} = \frac{d\,f(x^2)}{dx^2} \times (2x)$$

$$g'(2) = \left.\frac{d\,f(x^2)}{dx^2}\right|_{x=2} \times (2x)\big|_{x=2} = f'(2^2) \times (2 \times 2) = 4f'(4) \cong 4\left(\frac{5}{2}\right) \cong 10$$

Note: The derivative $f'(4)$ is estimated by placing a transparent plastic ruler in the position of tangent line at $(4, f(4))$ and computing the slope. Thus, the tangent at $(4, f(4))$ appears to pass through (3,0) and (5,5), so the "rise" = (5 - 0) = 5, "run" = (5 - 3) = 2, so $f'(2) \cong 5/2$.

Name: ______________________ Name: ______________________

Name: ______________________ Name: ______________________

As always, you must show your work and explain your reasoning. Failure to do so means few, if any, marks will be given.

1. a. Each of you should construct your own individualized fifth degree polynomial based on your student number according to the following scheme:

   i. Suppose your student number is 0234567.
   ii. We discard the leading 0, separate the remaining six digits, and alternate their signs: 2, -3, 4, -5, 6, -7.
   iii. We use the list of digits as coefficients to construct our polynomial: $f(x) = 2x^5 - 3x^4 + 4x^3 - 5x^2 + 6x - 7$

Solution:

b. Use either a graphing calculator, a LiveMath insert, or some other computer software to generate graphs (print and attach the graphs to your assignment). At least one graph must show the overall shape of your polynomial function, as well as the number of roots. Another graph must zoom in on a specific root (your choice if there is more than one), so that we can see the x-location of the root to 2 decimal place accuracy (that is, we can read tick marks on the x-axis that trap the root in an interval of length 0.001).

Solution: (Attach sheets)

c. Use a graphing calculator, a LiveMath insert, or some other computer software to generate three searches for the same root that you zoomed in on in 2. above using the three methods in Section 5 of Module 4, Bisection, False Position and Newton's Method. Use sheets posted in the Materials section for recording your searches. Start Bisection and False Position using your interval end points from your best graph in 2. Start Newton's Method with the midpoint of that same interval. Continue each search until one of the following conditions is true:

$|\text{error}| < 10^{-8}$ or $|f(c)| < 10^{-8}$ or the sheet is full

Solutions: (Attach sheets)

Bonus Question

d. With your starting interval, using logic and algebra, determine the smallest possible number of steps (iterations) required in the Bisection Method to have an accuracy such that $|\text{error}| < 0.5 \times 10^{-16}$

Solution:

Function: ______________________ Decimal Accuracy: __________

Iteration	x	f(x)	+/−	|error|
a				
b				
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
12				
14				
15				
16				
17				
18				
19				
20				
21				
22				
23				

Function: ______________________ Decimal Accuracy: ____________

Function Cut-off Value: ____________

Iteration	x	f(x)	+/-	|error|
a				
b				
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
12				
14				
15				
16				
17				
18				
19				
20				
21				
22				

Function: ______________________ Decimal Accuracy: __________

Function Cut-off Value: __________

Iteration	x	$f(x)$	+/-	|error|
a				
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
12				
14				
15				
16				
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