STANDING WAVES: A PHASE DIFFERENCE APPROACH

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ABSTRACT

A treatment of standing waves in one dimensional systems is presented based on an analysis of the phase difference between the direct wave and the reflected wave. The analysis leads to all the properties of the standing waves, in particular, the internodal distance, normal modes of vibration, the amplitude of the resultant wave as a function of position, and the role of multiple reflections from the boundaries of the medium.

§ 1. INTRODUCTION

Standing waves are formed in finite systems by interference between the direct wave and the wave reflected from the boundary. A common example is that of harmonic standing waves in one-dimensional systems such as a string fixed at both ends or an air column in a pipe. For determining an interference pattern, the phase difference approach is mathematically the simplest and intuitively the most appealing. Accordingly, standard texts use this approach for deducing interference patterns in a wide variety of situations except, unfortunately, for standing waves. The standard treatment is based on the resultant amplitude A as a function of position x for a string fixed at both ends calculated by superposing two waves with displacements given by $A_0 \sin(kx - wt)$ and $A_0 \sin(kx + wt)$. While $A(x)(= 2A_0 \sin kx)$, thus obtained, gives the correct internodal distance and (combined with the boundary conditions) also the correct normal modes, students often wonder why there is no reference to the phase difference (either due to the path difference, or due to reflection at the boundary) between the interfering waves, a factor which they have been told plays a crucial role in interference. An equally crucial point is the omission of any reference to the finite size of the medium [which only enters the calculation of the normal modes but not of A(x)]. All that seems to matter for the formation of standing waves is that the interfering waves be travelling in opposite directions. The treatment presented here is based on the familiar approach of the calculation of the phase difference between the interfering waves, involving the finite size of the medium at the very outset (§ 2). Since the phase difference between adjacent maxima (or adjacent minima) must be 2p radians, the expression for the phase difference immediately gives the internodal distance (§ 3) without any reference to A(x). Next, in § 4, we consider three systems:

(a) a string fixed at both ends
(b) an air column in a pipe open at both ends
(c) an air column in a pipe closed at one end and open at the other.

By combining the phase differences with the boundary conditions we calculate the normal modes. This is followed by the calculation of A(x). It would be seen that the calculation of normal modes must precede the calculation of A(x) because the phase difference depends not only on x but also on ℓ, the length of the medium. Then, in § 5, we discuss the effect of multiple reflections from the boundaries of the medium. Finally, § 6 contains some concluding remarks.

§ 2. PHASE DIFFERENCE BETWEEN THE DIRECT WAVE AND THE REFLECTED WAVE

<———— the medium ————>
x = 0 x = ℓ
<——x——>
|————x————————|$_B$

————> direct wave
<————
reflected wave

Suppose the direct wave is represented by

$$y_1(x,t) = A_1 \sin(kx - \omega t). \qquad (1)$$

At any point x the reflected wave, interfering with the direct wave, will have the following features:

(i) The reflected wave travels towards the negative x direction whereas the direct wave travels towards positive x.
(ii) The angular frequency ω and the wave number k are the same for the direct wave and the reflected wave.

(iii) Before arriving at the interference point x, the re flected wave will have travelled an extra distance $2(\ell - x)$.

(iv) The reflected wave will have suffered a phase change ϕ_r due to reflection at the boundary B. The value of ϕ_r is either zero or π radians, depending on the nature of the boundary. For each of the three systems, mentioned in § 1, ϕ_r would be given its appropriate value before analyzing the normal modes and A(x) in § 4.

All of the features (i) to (iv) are incorporated into the reflected wave represented by[1]

$$y_2(x,t) = A_2 \sin[kx + 2k(\ell - x) - \omega t + \phi_r]. \qquad (2)$$

Thus, the phase difference ϕ between the two interfering waves can be written as

$$\phi = \phi(x) = \phi_p(x) + \phi_r, \qquad (3)$$

where ϕ_p is the phase difference due to path difference, given by

$$\phi_p = 2k(\ell - x). \qquad (4)$$

§ 3. CALCULATION OF THE INTERNODAL DISTANCE

In terms of A_1, A_2 and f, the resultant amplitude A is given by

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos \phi} \qquad (5)$$

Of special interest are points of maximum and minimum values of A which, according to (5), are given by

$$\text{MAXIMA} \quad |\phi| = 0, 2\pi, 4\pi \ldots\ldots \qquad (6)$$
$$\text{MINIMA} \quad |\phi| = \pi, 3\pi, 5\pi \ldots\ldots \qquad (7)$$

In the context of standing waves, minima are called nodes[2] (N), and maxima are called antinodes (A_N).

Since ϕ_r in (3) is independent of x, combining (6) and (7) with (4) we find that the distance D between two adjacent antinodes is equal to the distance between two adjacent nodes, each being given by

$$2p = \Delta\phi = \Delta\phi_p = 2kD \qquad (8)$$

Using $k = \frac{2\pi}{\lambda}$, from here we get

$$D = \lambda/2 \qquad (9)$$

Similarly, the distance d between a node and an adjacent antinode is given by

$$\pi = \Delta\phi = 2kd,$$

$$\text{or, } d = \lambda/4. \qquad (10)$$

§ 4. NORMAL MODES OF VIBRATION

Normal modes of vibration are determined by using the conditions imposed by the boundary. For a one-dimensional system, at one of the two boundaries, the required condition is automatically satisfied by the appropriate value of ϕ_r (which is how ϕ_r is determined in the first place). At the other boundary, the required condition is satisfied only by imposing a restriction on the permitted wavelengths (or frequencies). Wavelengths which obey the restriction are called normal modes of vibration.

As mentioned in § 1, we will determine the normal modes for three systems. In each case we will also calculate the resultant amplitude A as a function of x.

(a) String fixed at both ends

In this case ϕ_r is π. So, from (3) and (4) we get

$$\phi = 2k(\ell - x) + \pi \qquad (11)$$

Using (11), in (6) and (7) we get

$$\text{MAXIMA} \quad 2k(\ell - x) = \pi, 3\pi, 5\pi \ldots\ldots \qquad (12)$$
$$\text{MINIMA} \quad 2k(\ell - x) = 0, 2\pi, 4\pi \ldots\ldots \qquad (13)$$

We require both boundaries $x = 0$ and $x = \ell$ to be nodes. From (13), the boundary condition at $x = \ell$ is seen to be satisfied automatically (by virtue of $\phi_r = \pi$). For $x = 0$ to be a node, (13) requires[3]

$$2k\ell = 2\pi, 4\pi, 6\pi \ldots\ldots, \qquad (14)$$

which gives $\ell = n \frac{\lambda}{2}$ $(n = 1, 2, 3\ldots)$. (15)

For a given ℓ, the restriction (15) on the permitted wavelengths gives the normal modes of this system.

1 Note that by rewriting (2) as $y_2(x,t) = A_2 \sin(2k\ell - kx - \omega t + \phi_r)$, we can see more clearly that the wave represented by (2) is travelling in the negative x-direction.

2 It is evident from (5) that the amplitude at a node is zero only if $A_1 = A_2$.

3 Note that $2k\ell = 0$ is not possible.

Now, to determine A as function of x, we use (11) and (14) in (5) whence we obtain

$$A(x) = \sqrt{A_1^2 + A_2^2 - 2A_1A_2 \cos 2kx} \qquad (16)$$

If $A_1 = A_2 = A_0$, (16) takes a simpler and a more familiar form (using the trigonometric identity: $1 - \cos 2\theta = 2 \sin^2\theta$)

$$A(x) = 2A_0 \mid \sin kx \mid \qquad (17)$$

Incidentally, using (11) and (14) in (2) we also notice that the reflected wave is indeed correctly given by

$$y_2(x,t) = A_2 \sin(kx + \omega t); \qquad (18)$$

the form used in the standard treatment of standing waves.

However, in order to arrive at (18) we must use the condition (14) for the normal modes, rather than the approach used in the standard texts where (18) is used in order to derive A(x) and whence at the normal modes.

(b) Air column in a pipe open at both ends

In this case $\phi_r = 0$ which gives

$$\phi = 2k(\ell - x). \qquad (19)$$

Both boundaries are required to be antinodes, governed by (6). At $x = \ell$, the requirement of the antinode is automatically satisfied (by virtue of $\phi_r = 0$). At x = 0 the requirement of the antinode once again leads to (14) and (15) so that the normal modes of this system are the same as those of (a). However, since (11) of (a) is different from (19) of (b), (16) and (17) for A(x) are replaced, respectively, by

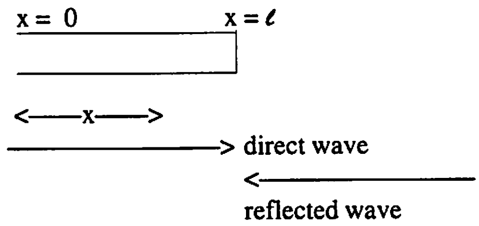
$$A(x) = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos 2kx} \qquad (20)$$
$$= 2A_0 \mid \cos kx \mid \text{ if } A_1 = A_2 = A_0 \qquad (21)$$

Also, (18) is replaced by

$$y_2(x,t) = -A_2 \sin(kx + \omega t). \qquad (22)$$

Note that in this case the sign of y_2 is opposite to that used in the standard treatment of standing waves.

(c) Air column in a pipe open at one end and closed at the other



In this case $\phi_r = \pi$ so that (11), (12) and (13) of (a) apply also to (c). Furthermore, as in (a), the requirement that the boundary $x = \ell$ be a node is automatically satisfied by virtue of $\phi_r = \pi$. However, since now the boundary x = 0 is required to be an antinode (whereas in (a), x = 0 was a node), the normal nodes in (c) are obtained with the help of (12):

$$2k\ell = \pi, 3\pi, 5\pi \ldots\ldots \qquad (23)$$

or, $$\ell = n \frac{\lambda}{4} \quad (n = 1, 3, 5...) \quad \frac{\lambda}{4} \qquad (24)$$

Using (11) and (23) in (5) we find that (20) and (21) for A(x) and (22) for $y_2(x,t)$, found in (b), also apply to (c).

§ 5. THE EFFECT OF MULTIPLE REFLECTIONS

If we denote the direct wave by W and the successive reflected waves by R_1, R_2, R_3 ... (where R_n denotes a wave which has undergone n reflections), the resultant wave is obtained by the superposing W, R_1, R_2, R_3 ... .
The result of this superposition can be seen easily by rearranging the different contributions as $(W + R_1) + (R_2 + R_3) + (R_4 + R_5) + \ldots\ldots$ and by noting that each successive term in this sum is in phase[4] with the previous one. Thus, the resultant amplitude is simply the sum of the amplitudes of $(W + R_1)$, $(R_2 + R_3)$, $(R_4 + R_5)$ etc., i.e., multiple reflections only reinforce the interference pattern produced

4 This point is intuitively obvious but can also be proven in detail as follows:
By following the same argument as in § 2 for f_{WR} (which stands for the phase difference between W and R_1), we can see that

$$\phi_{WR_2} = \phi_{R_2R_4} = \phi_{R_4R_6} \ldots\ldots$$
$$= \phi_{R_1R_3} = \phi_{R_3R_5} = \phi_{R_5R_7}$$
$$= 2k\ell + \phi_r + \phi'_r,$$

where ϕ_r' is the phase change on reflections at the boundary x = 0.
In (a) and (b), $2k\ell$ = even multiple of p. In (a), $\phi_r = \phi'_r = \pi$; in (b), $\phi_r = \phi'_r = 0$. In (c), $2k\ell$ = odd multiple of π; $\phi_r = \pi$ and $\phi_r' = 0$. Thus, in (a), (b) as well as (c), we find that $(W, R_2, R_4 \ldots)$ are in phase, and so are $(R_1, R_3, R_5 \ldots)$.

by $(W + R_1)$. Of course, each reflection being partial, the amplitude of successive reflections becomes progressively smaller.

§ 6. CONCLUDING REMARKS

We have shown that all aspects of standing waves in one-dimensional systems can be deduced on the basis of the phase difference between the direct wave and the reflected wave. The treatment also brings out more clearly the role played by the finite size of the medium, an element crucial to the formation of standing waves. As an added bonus, we are also able to see that the effect of the ever-present multiple reflections is simply to reinforce the interference pattern formed by the direct wave W and the wave R_1 which suffers only one reflection.

The approach used in the standard texts is inadequate in many respects, including a complete lack of justification for writing the reflected wave as $A_0 \sin(kx + \omega t)$, for the string and $-A_0\sin(kx + \omega t)$ for the air-column. Students already use the phase difference approach in numerous examples of interference such as double slit, thin films, interferometers, diffraction grating etc. So, why not use the same for standing waves, especially in view of the simplicity of the accompanying mathematics?

ACKNOWLEDGEMENT

The author gratefully acknowledges a helpful discussion with Dr. Gerald Portner.